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A COMBINED APPROACH TO SOLVE BINARY MULTICRITERIA PROBLEMS*

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ABSTRACT

An implicit enumeration algorithm is developed to determine the set of efficient points in zero-one multiple criteria problems. The algorithm is specialized for the solution of a particular class of facility location problems. The procedure is complemented with the use of the utility function of the decision maker to identify a subset of efficient point candidates for the final selection. Computational results are provided and discussed.

1. INTRODUCTION

The zero-one linear multiple criteria problem (MCP) is written as

$$(P) \quad \text{Max}\{Cx : x \in F\}$$

where $F = \{x \in R^n : Ax \leqslant b,\ x_j = 0 \text{ or } 1,\ j = 1, 2, \ldots, n\}$, C and A are respectively $p \times n$ and $m \times n$ real matrices, b is an m-vector and x is an n-vector. The solution set to (P) is defined as

$$EF = \{x^0 \in F \text{ such that } x \in F \text{ and } Cx \geqslant Cx^0 \text{ implies } Cx = Cx^0\}.$$

The elements of EF are said to be the efficient points in (P). We also refer to EF as the efficient set.

The continuous linear multiple objective problem has the property that every efficient point maximizes a linear functional, of the form λCx with $\lambda \in R^p$ and $\lambda > 0$, over the feasible set. This property does not hold for problem (P) and makes its solution considerably more difficult. As a consequence, the obtainment of EF must rely on algorithms based on different principles than those developed to solve the continuous linear multiple criteria problem.

Pasternak and Passy [6] studied problem (P) for the case of two critieria. They presented an algorithm based on parametric approach combined with an extension of Balas' filter method. Pasternak and Passy applied the algorithm to the project selection problem and reported the results in [5]. One of the authors developed in [1] and [2] serveral theoretical results and two algorithms to solve (P). The algorithm in [2] explores the fact that the set of efficient points in (P) can be seen as the set of isolated and end nodes of a related directed graph. In the same paper the author studies the version of (P) where the set F is given explicitly as a finite discrete set. This last problem was also analyzed by Zionts and Wallenius [14] and Zionts [13]. Shapiro

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[8] mentions several applications for problem (P) and relates it to integer programming duality theory. Villarreal and Karwan [11], [10] presented a dynamic programming approach to solve problem (P) for the case of integer variables not necessarily restricted to be zero-one. More recently Bitran and Lawrence [3] applied the algorithm developed in [2] to solve a facility location problem formulated as a zero-one multiple criteria problem.

The determination of the set EF can be seen as a first step in solving the decision problem implied by (P). Typically, the decision maker wants to select just one point and therefore a second step must be undertaken to select such point. An alternative approach to the one taken in the references cited above is the utility theory methodology [4]. These two classes of methods have been developed almost separately although they are both interested in the same types of problems. We believe that both approaches have their weaknesses. The efficient point methods fall short of solving the decision problem, while the utility theory procedures rely on the assumption that the decision maker's utility function can be obtained with satisfactory accuracy. Whenever this assumption does not hold, which seems often to be the case in practical applications, the analyst should perform an extensive sensitivity analysis on the coefficients of the utility function. However, sensitivity analysis for integer programs can be quite complex and may require that the problem be resolved for each possible value of the parameters.

In this paper we first propose a new algorithm to determine the set of efficient points in binary multiple criteria problems and next we indicate how utility theory can be used effectively to select a solution point. It is well known that if the decision maker's utility function is nondecreasing in each of the attributes, i.e., criterion, there is at least one element of EF that maximizes the utility function over F. Based on this result, we propose that first EF be determined and then the utility function be assessed and evaluated at each efficient point. The advantage of this approach is that since the set EF is finite we can perform an extensive sensitivity analysis on the uncertain coefficients of the utility function in a very simple way. Also, the range of variation of each attribute is precisely known prior to the determination of the utility function.

The plan of this paper is as follows. In section 2 we develop an implicit enumeration scheme to determine the set EF. In section 3 we report the results of computational experiments on the performance of the algorithm and illustrate how the implicit enumeration scheme can be complemented with the use of the utility theory of the decision maker. In the last section we provide a comparison of the algorithm with those of Villarreal and Karwan [11] and Bitran [2], and discuss conclusions.

2. AN IMPLICIT ENUMERATION ALGORITHM TO OBTAIN EF

The algorithm is composed of the following main subroutines: Finding Bounds for the Criteria, Fathoming by Bounds, Testing Feasibility and Determining Bounds for the Variables, Backtracking, Branching, and Dominance. These routines are described in this section. After having set k variables at the value zero or one, we are left with problem (P_k) that we write as

$$(P_k) \quad \text{Max}\left\{ \sum_{j \in J - R_k} c_{ij} x_j + \sum_{j \in R_k} c_{ij} x_j \quad i = 1, 2, \ldots, p \right\}$$

$$\text{Subject to} \sum_{j \in J - R_k} a_{ij} x_j \leqslant S_{ik} \quad i = 1, 2, \ldots, m \tag{2.1}$$

$$x_j = 0, 1 \; j \in J - R_k. \tag{2.2}$$

The set R_k is the set of indices of the variables previously set at values one or zero, J is the set $\{1, 2, \ldots, n\}$, $J - R_k$ is the set of indices of the free variables, and $S_{ik} = b_i - \sum_{j \in R_k} a_{ij}x_j$ $i = 1, 2, \ldots, m$.

The block diagram representing the algorithm is given in Figure 1. The initial lower bounds (discussed later in this section) and some of the feasible points determined by the algorithm are stored in a matrix denoted by LB. Each row of LB contains the n-vector x^i and the corresponding p-vector Cx^i. After passing by the Dominance routine, the elements in LB are the efficient points in (P). The indicator LEV (level in the tree) specifies the number of variables already fixed at value zero or one. It is convenient to introduce the following notation. For $x^1, x^2 \in R^n$, $Cx^2 > Cx^1$ mean $Cx^2 \geqslant Cx^1$ with at least one strict inequality. $x^1 \in R^n$ is said to be dominated by $x^2 \in R^n$ if $Cx^2 \geqslant Cx^1$. In this case we also say that x^2 dominates x^1.

2.1 Finding Bounds for the Criteria, Fathoming By Bounds, and Inverse Dominance

(These three routines are intimately related and are described jointly.)

Among the first authors to use bounds for the criteria in multiple objective problems are Villarreal and Karwan [11]. In their combined dynamic and branch and bound approach, they define a lower bound for the criteria as a p-vector $z^0 = Cx^0$, where x^0 is either efficient or it is dominated by at least one efficient solution of the problem. A lower bound z^0 is used for fathoming purposes as follows: If for some k, the p-vector $z^k_{\max}$, composed of the maximum of each criterion in (P_k), is dominated by z^0, i.e., $z^0 \geqslant z^k_{\max}$, (P_k) can be fathomed. The reason is that for any feasible point x in (P_k), the relation $z^0 \geqslant Cx$ will hold and consequently $x \notin EF$. $z^k_{\max}$ is found by adding to the p-vector of criteria computed with the fixed variables at stage k, z^k, an upper bound on the criteria, $UB(n-k)$, computed with the free variables. $UB(n-k)$ is obtained by setting for every objective i each free variable j at value one or zero, depending upon the sign of its coefficient c_{ij}. x_{ij} is set equal to one if $c_{ij} > 0$ and to zero otherwise. The components of $z^k_{\max}$ are upper bounds for the corresponding criterion in (P).

Lower bounds can be generated by several methods. One consists in solving a relaxed version of (P) obtained by ignoring the constraints $Ax \leqq b$ and next selecting those efficient points that satisfy the constraints initially deleted. Note that this set is a subset of EF. Another method, used by Villarreal and Karwan [11], maximizes a positive combination of the criteria, λCx with $\lambda > 0$, over the feasible set F. The points obtained are efficient in (P). In fact, any feasible point x in F generates a p-vector Cx that can be used as a lower bound. However, those determined by points $x \in EF$ are the most effective. In out algorithm we initially generate lower bounds by a variant of the last method. A vector $\lambda \in R^p$, $\lambda > 0$ is chosen (see Section 3) and the vector λC is computed. A variable with the highest positive coefficient in λC is set equal to one. Next, a variable with the remaining highest positive coefficient in λC is set equal to one and we continue in this fashion until either all variables with positive coefficients are set to the value one or we are prevented from continuing because we have reached an infeasible solution. We illustrate this procedure with an example.

EXAMPLE 1: Assume that $\lambda C = (3, 2, 1, -2, -1)$ and

$$F = \{x \in R^5 : x_1 + x_2 + 5x_3 + x_4 + 3x_5 \leqslant 6, \quad x_j = 0, 1 \quad j = 1, 2, 3, 4, 5\}.$$

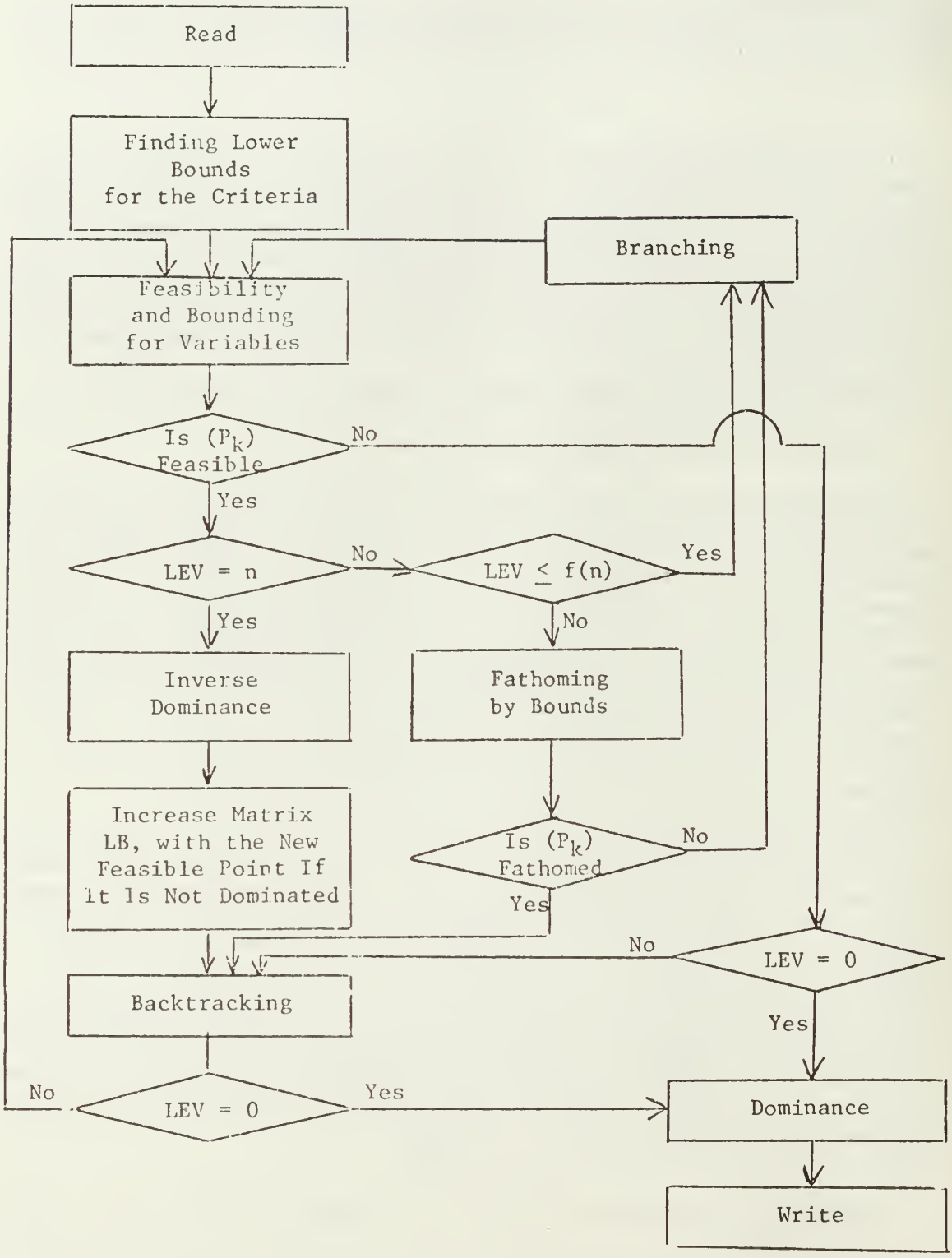

FIGURE 1. An overview of the algorithm

Thus, x_1 and x_2 are set equal to one and all other variables are set equal to zero. If we denote the feasible solution obtained by x^0, Cx^0 is used as a lower bound. Early experiments performed to test the effectiveness of some methods to generate lower bounds suggested that the variant used presents a satisfactory tradeoff between the quality of the bound and the time spent to compute it.

For small values of k the values of the upper bounds $z^k_{\max}$ are fairly "large" and the bounding of the criteria is not effective. For this reason we have explored the alternative of using the subroutine Fathoming by Bounds for $k \geqq f(n)$ where $f(n)$ is a pre-specified number. The computational results obtained varying $f(n)$ are presented in the next section.

Whenever we reach the end of a branch in the tree enumeration method ($LEV = n$) and find a point x^1 (note that x^1 is such that $Ax^1 \leqq b$ and $Cx^1 \geqslant z^0$). We compare it with the elements in matrix LB. If some element x^i of LB dominates $x^1 (Cx^i \geqslant Cx^1)$, we discard x^1. Otherwise, it is simply added to the matrix. We call this test Inverse Dominance. The experiments performed indicated that it is not worthwhile to also test if the feasible point dominates elements in matrix LB. Rather, it is more effective to execute the subroutine Dominance at the end of the algorithm.

2.2 Feasibility and Bounding for Variables

The extended Geometric Definition Method of Zionts [12] is applied in this paper for the multicriteria case. The objective is to find upper and lower bounds on some variables, those most likely to be bounded. If for some variable j, the upper bound u_j is less than one, the variable is set at value zero and if a lower bound h_j greater than zero is found, the variable is set at value one. The bounds are found by setting the remaining variables at zero or one according to the sign of their coefficients in the constraint matrix A.

The block diagram of Figure 2 illustrates the subroutine Testing Feasibility and Determining Bounds for the Variables. Each constraint i is considered differently, depending upon the sign of its right-hand-side (RHS) S^k_i at stage k.

If the RHS is negative, we test to see if $S^k_i < SAN^k_i$ (SAN^k_i is the sum of the negative coefficients of the free variables of constraint i at stage k). In this case (P_k) is infeasible and the algorithm backtracks. If $S^k_i = SAN^k_i$, all negative variables are set at one and all positive variables at zero. Finally, if $S^k_i > SAN^k_i$ the lower bound h_j is computed for a variable with the most negative coefficient and the upper bound u_j is computed for a variable with the most positive coefficient.

If the RHS is positive, we test to see if constraint i satisfies the condition $S^k_i \geqq SAN^k_i + SAP^k_i$ where SAP^k_i is the sum of all positive coefficients of the free variables of row i at stage k. In that case we continue to the next constraint. Otherwise, we find the upper bound for a most positive variable.

This simplified method of finding bounds for only the most positive and most negative variables of every row is analyzed in Rivera [7]. A procedure for finding all the variables that need to be tested for bounding is also discussed.

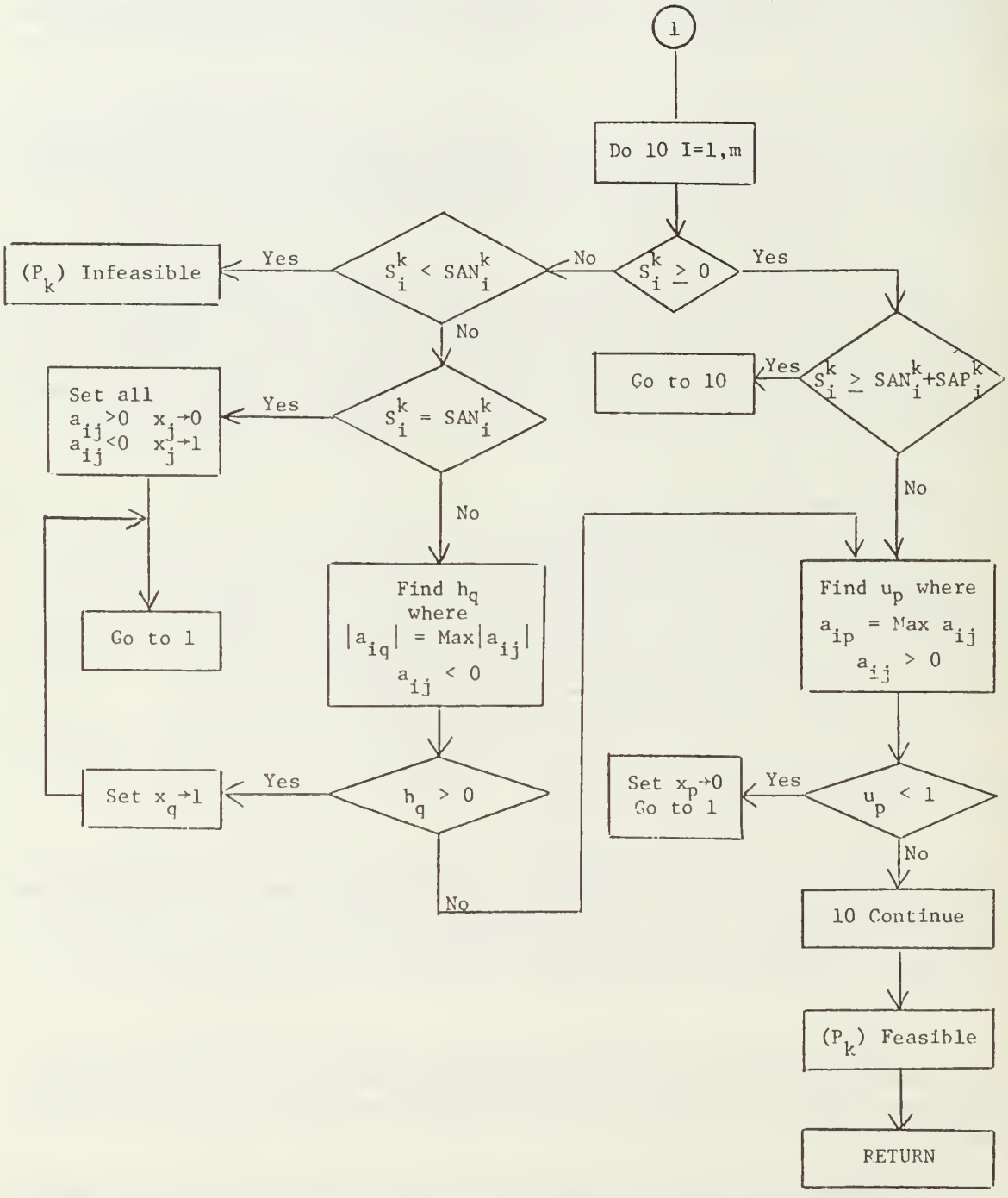

FIGURE 2. Subroutine: testing feasibility and determining bounds for the variables

2.3 Backtracking

The backtracking subroutine is very simple. The algorithm moves backward in the tree until a variable which has taken only one of the two possible values is found. The variable is set at the other value and the algorithm proceeds as in Figure 1. Variables whose values are directly set in the Feasibility and Bounding for Variables subroutine are skipped over in the backtracking procedure since their complementary values are not worth investigating.

2.4 Branching

The branching rules are designed to drive the problem towards infeasible parts of the tree so that the subroutine Feasibility and Bounding for Variables just described is more effective.

The block diagram of Figure 3 shows the branching rule strategy. Only one constraint is analyzed each time. If there are some constraints with negative RHS's, the algorithm chooses one with $\underset{i}{\text{Min}}\,(S_i^k - SAN_i^k)$ and branches on a variable x_q with most negative coefficient, setting x_q at value zero.

In the absence of constraints with negative RHS, the algorithm attempts to create one. In this case a constraint with the least slack, $\underset{i}{\text{Min}}\,(S_i^k - SAN_i^k - SAP_i^k)$, is chosen and a variable with the largest positive coefficient is set equal to one.

2.5 Dominance

Before entering the Dominance subroutine, matrix LB contains the lower bound used. These include the initial lower bound determined and all feasible points not eliminated by the Inverse Dominance subroutine. The set of efficient points in (P) is contained in the set of elements of LB. Therefore, LB may contain dominated points. In order to identify these point, pair-wise comparisons between all elements in LB were performed.

Another procedure tested was as follows. The elements in LB were kept ordered by decreasing value of the sum of their components. Each feasible point x^1 determined by the algorithm was tested to see if it dominated or was dominated by elements in LB. We have used the fact that a necessary condition for x to dominate y is

$$\sum_{i=1}^{p}\sum_{j=1}^{n} c_{ij}x_{ij} > \sum_{i=1}^{p}\sum_{j=1}^{n} c_{ij}y_{ij}.$$

Although with this procedure, at the end of the algorithm, matrix LB contained the efficient points of the problem (without the need for the Dominance subroutine), it did not improve the computation time.

2.6 A Class of Facility Location Problems

We conclude this section specializing the implicit enumeration algorithm for a class of facility location problems of the form

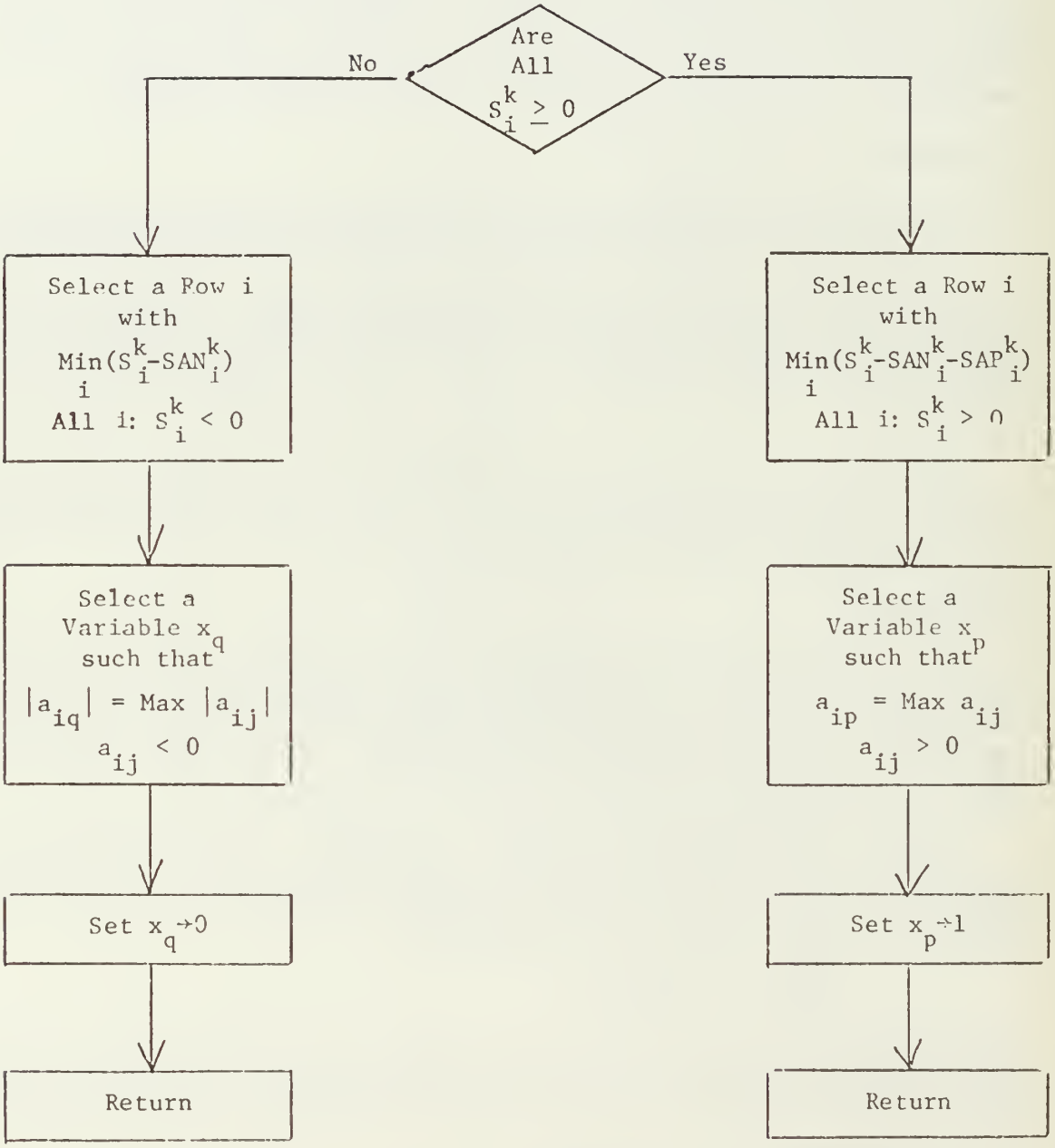

FIGURE 3. Subroutine: branching

$$\text{(FLP)} \quad \text{Max } Cx$$
$$\text{subject to } Ax \leqq b$$

$$\sum_{j \in J_r} x_j \leqq 1 \quad r = 1, 2, \ldots, q \tag{2.3}$$
$$x_j = 0 \text{ or } 1 \quad j = 1, 2, \ldots, n$$

where $J_1, J_2, \ldots, J_q$ form a partition of $J = \{1, 2, \ldots, n\}$ and C, A, b, and x are as in (P). The constraints (2.3) state that at most one element of each set J_r, $r = 1, 2, \ldots, q$ can be part of an efficient point of (FLP). Such conditions are common in practical settings, mainly when the sets J_r represent heterogeneous geographical regions with laws and regulations that differ significantly. Bitran and Lawrence [3] report an application of (FLP). The special structure of constraints (2.3) permits a significant simplication in the branch and bound algorithm described.

In (FLP) a maximum of q variables can be set to one. This fact is exploited in the Fathoming by Bounds subroutine. When $UB(J - k)$ is determined, it is not necessary to set all the positive free variables to one, but only the $q - w$ with the largest positive coefficients, where w is the number of variables fixed at value one. We also modified subroutines Feasibility and Bounding for Variables and Fathoming by Bounds to exploit the fact that at most one variable among those in which indices are elements of each J_r may be one.

The obtainment of lower bounds for the criteria is also slightly changed for this special structure. After computing the vector λC, $\lambda > 0$, $\lambda \in R^p$, the algorithm chooses a variable in J_1, the one with the largest positive coefficient in λC and sets it equal to one. Next, the variable in J_2 with the largest positive coefficient in λC is set equal to one. The procedure continues in this fashion until either one variable for every J_r, $r = 1, 2, \ldots, q$, is set to the value one or an infeasibility is reached.

3. COMPUTATIONAL RESULTS

In this section we present computational results on the performance of the implicit enumeration algorithm of Section 2. We have attempted to analyze the impact of the following parameters, the number of lower bounds, number of objective functions, magnitude of the RHS, number of variables, and the stage in which to start the bounding of the objectives. Both problems (P) and (FLP) are studied in these experiments. We conclude this section by illustrating the combination of the efficient points and utility theory approaches.

All problems tested were randomly generated with the coefficients in matrices A and C in the interval [0,100]. The RHS of each constraint was set at a fixed proportion K ($K = .25, .50, .75$) of the sum of the corresponding coefficients in matrix A. The reason for this selection of data is to permit comparison with other works in the liturature [1], [2], and [11]. The results obtained with the special structure of the particular facility location problem (FLP) are significantly superior to the corresponding results for problem (P). All computations were made with the computer PRIME 400. The computer programs were written in FORTRAN.

The following additional notations will be used:

NTB = number of problems (P_k) fathomed for $k = 1, 2, \ldots, n$.

NPA = number of feasible points after bounding (i.e., number of end branch points).

NEP = number of efficient points in (P).

Q = number of feasible points after inverse dominance.

T = Total central processing unit (c.p.u.) time in seconds.

BT = c.p.u. seconds spent in bounding the objectives.

3.1 Problem (P)

The impact of the number of lower bounds and the stage at which to start the bounding procedure were initially explored.

As did Villarreal and Karwan [11] for their dynamic programming algorithm, we tested several problems with one, two, and three lower bounds. The values of $\lambda \in R^3$ chosen were (1,1,1), (3,1,1) and (1,1,3). The results were highly dependent upon the size of the problem. For problems with up to fourteen variables, one lower bound performed better; for fifteen to twenty variables, two lower bounds gave the best results; and for more than twenty variables, three lower bounds showed superior performance. In Table 1 we present a sample of the experiments performed for problems of type (P). The c.p.u. times correspond to the mean and standard deviation of a sample of five problems.

TABLE 1 — *Problem (P), c.p.u. Times, $p = 3$, $n = 14$, and $m = 4$*

		One Lower Bound	Two Lower Bounds
$K = .75$	Median	33.12	41.04
	Range	25.20-66.60	28.44-67.68
$K = .50$	Median	35.64	56.88
	Range	18.72-136.44	42.48-163.44
$K = .25$	Median	15.12	19.08
	Range	11.88-31.68	14.76-34.92

In Figures 4a and 4b we present a sample of typical results obtained when the bounding procedure starts at different levels, i.e., after having fixed a certain number of variables. The computational experiments indicated that the bounding procedure was not effective when introduced at levels smaller than $n - 3$, i.e., $f(n) < n - 3$. Figures 4a and 4b illustrate the case

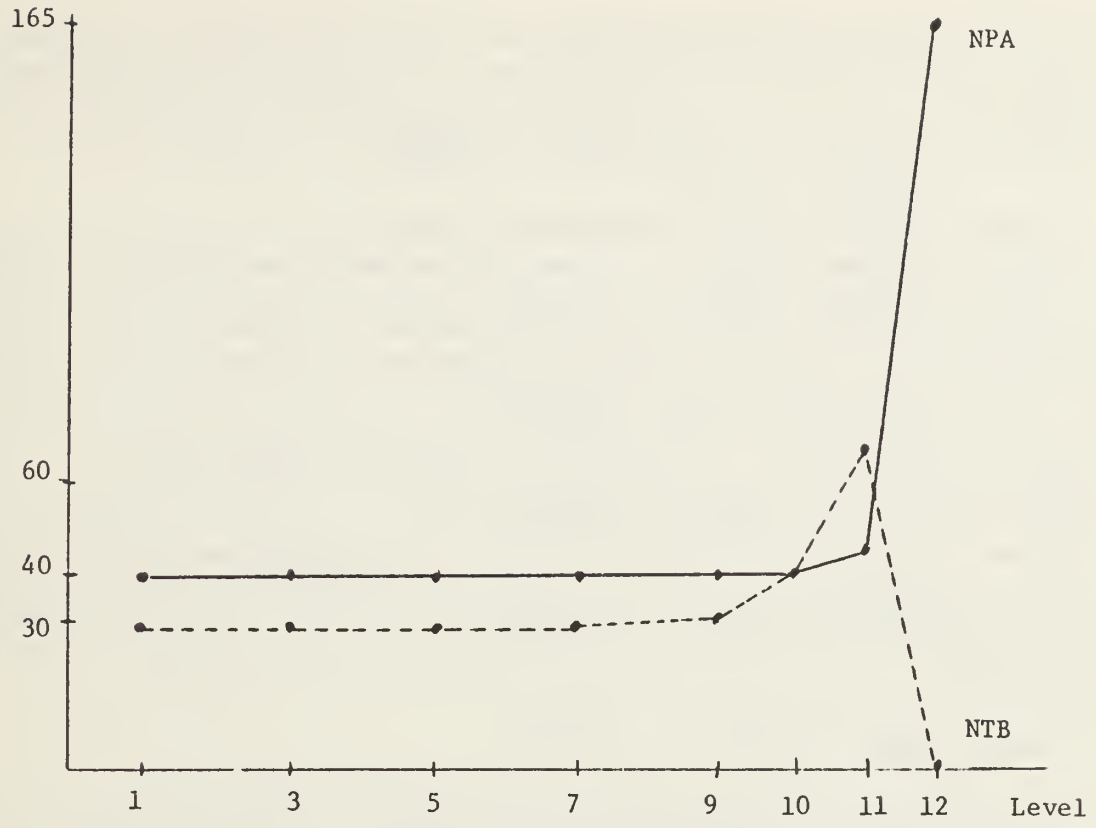

FIGURE 4a.

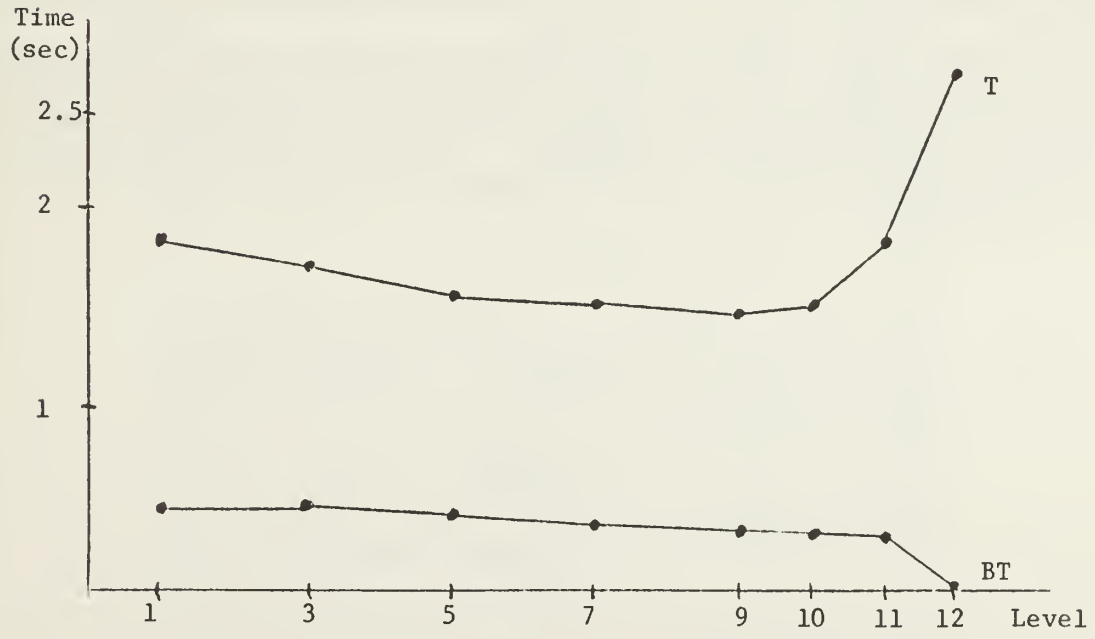

FIGURE 4b.

FIGURE 4. The effect of varying the bounding level — $p=3$, $n=12$, $m=6$, $K=.25$

where the bounding procedure was started at each of the levels indicated. The decrease in computation time between level 1 and level 9 represent the savings for not using the bounding procedure. For $f(n) > n - 3$, NTB and T start to increase because the bounding routine fathomed a large number of problems (P_k). The points in both figures for LEVEL = 12 correspond to not using the Fathoming by Bounds subroutine.

Tables 2 and 3 show the results obtained with problems of type (P) with varying numbers of objectives, variables and factor K. For each set of parameters, five problems were solved. The computational results suggest that increasing the number of objectives increases the computation time. Essentially, the reason is that the number of efficient points also increases. Problems with $K = .25$ tend to be easier to solve because the number of feasible points is lower than for $K = .50$ and $K = .75$. Problems with $K = .25$ are also favored by the design of the algorithm that branches toward infeasibility. By branching in the opposite way we can obtain, for problems with $K = .75$, essentially the same results observed with $K = .25$. The same effect occurs with other algorithms in the litureature [2]. Therefore the hardest problems to solve in terms of c.p.u. time are the problems with $K = .50$. An intuitive justification for these observations is the fact that for $K = .25$ [$K = .75$] most efficient points tend to be close to $(0, 0, \ldots, 0) \in R^n[(1, \ldots, ,1) \in R^n]$. To visualize the impact of the number of variables on the computation time, we have plotted in Figure 5 the mean c.p.u. time for the problems with $p = 3$, $m = 8$, $K = .25$ of Table 3. As expected, the computation time increases exponentially.

3.2 Facility Location Problems

In Table 4 we present computational results corresponding to problems of type (FLP). A sample of five problems was solved for each set of parameters. From the nine constraints, five are of the special structure $\sum_{x_j} \leqq 1$. The comments made for Problem (P) with different values of K also apply for the results in Table 4. However, the times to solve the facility location problems are considerably smaller than those needed for Problems (P). Of course the reason is that (FLP) has a special structure that permits a specialization of the algorithm. In Figure 6 the computation times obtained for (P) and (FLP) are compared. The solid line corresponds to the curve of Figure 5, while the dashed line represents the problem of Table 4 with $p = 3$, $m = 9$, and $K = .25$.

TABLE 2 — *Problem (P)*

		T(sec)	NEP
p=5, m=8, K=.25, n=10	Median	0.31	4
	Range	0.24-0.42	2-9
p=5, m=8, K=.25, n=14	Median	1.59	17
	Range	1.14-1.73	10-24
p=5, m=8, K=.25, n=18	Median	12.97	70
	Range	10.15-13.57	37-91

TABLE 3 — *Problem (P)*

		T(sec)	NEP
$p=3,\ m=8,\ K=.75,\ n=10$	Median	2.02	10
	Range	1.35-2.23	4-25
$p=3,\ m=8,\ K=.50,\ n=10$	Median	1.06	10
	Range	0.93-1.50	7-14
$p=3,\ m=8,\ K=.25,\ n=10$	Median	0.22	5
	Range	0.17-0.25	1-6
$p=3,\ m=8,\ K=.75,\ n=14$	Median	15.63	20
	Range	13.80-20.08	11-55
$p=3,\ m=8,\ K=.50,\ n=14$	Median	14.70	22
	Range	8.94-17.34	15-62
$p=3,\ m=8,\ K=.25,\ n=14$	Median	1.08	5
	Range	0.73-1.25	3-11
$p=3,\ m=8,\ K=.75,\ n=18$	Median	223	36
	Range	194-355	26-62
$p=3,\ m=8,\ K=.50,\ n=18$	Median	195	36
	Range	138-216	22-54
$p=3,\ m=8,\ K=.25,\ n=18$	Median	8.00	20
	Range	6.39-8.20	13-41
$p=3,\ m=8,\ K=.25,\ n=20$	Median	16.76	14
	Range	14.80-21.30	12-36
$p=3,\ m=8,\ K=.25,\ n=23$	Median	77.30	24
	Range	66.70-104.40	16-60
$p=3,\ m=8,\ K=.25,\ n=25$	Median	229	50
	Range	170-290	28-76

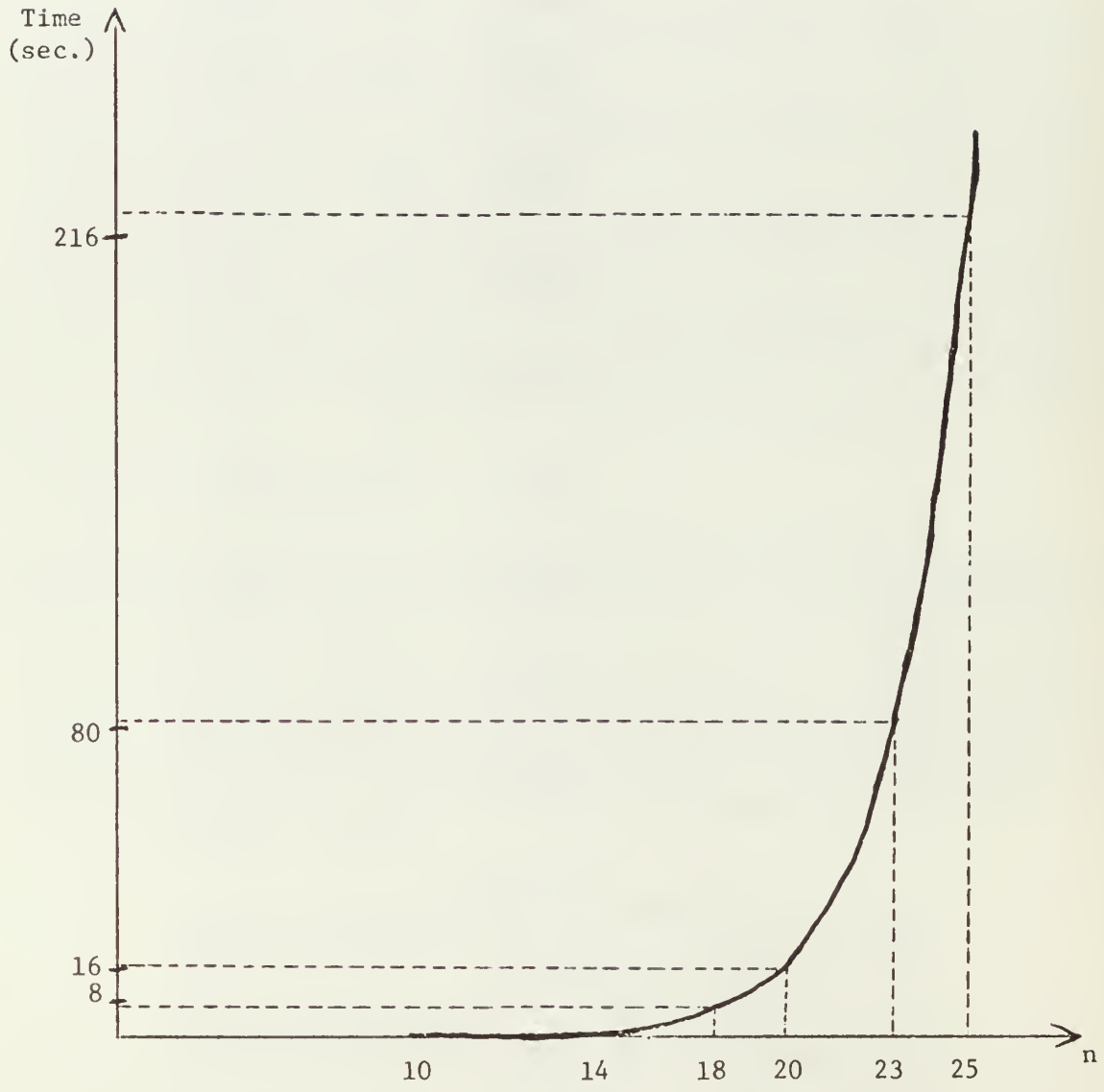

FIGURE 5. Data from Table 3 for problems with $p=3$, $m=8$, $K=.25$

TABLE 4 — *Problem (FLP)*

		T(sec)	NEP
$p=3,\ m=9,\ K=.75,\ n=10$	Median	1.91	14
	Range	1.28-2.11	8-22
$p=3,\ m=9,\ K=.50,\ n=10$	Median	1.83	8
	Range	0.98-2.07	1-12
$p=3,\ m=9,\ K=.25,\ n=10$	Median	0.58	5
	Range	0.55-0.71	3-8
$p=3,\ m=9,\ K=.75,\ n=15$	Median	5.61	28
	Range	3.08-7.05	8-46
$p=3,\ m=9,\ K=.50,\ n=15$	Median	7.17	28
	Range	3.33-9.39	8-47
$p=3,\ m=9,\ K=.25,\ n=15$	Median	3.78	7
	Range	3.45-3.86	4-11
$p=3,\ m=9,\ K=.75,\ n=20$	Median	14.93	42
	Range	7.51-30.16	13-51
$p=3,\ m=9,\ K=.50,\ n=20$	Median	13.60	42
	Range	7.35-32.35	13-44
$p=3,\ m=9,\ K=.25,\ n=20$	Median	13.18	23
	Range	10.51-19.22	10-48
$p=3,\ m=9,\ K=.25,\ n=25$	Median	37.83	58
	Range	23.95-66.75	20-80
$p=3,\ m=9,\ K=.25,\ n=30$	Median	111.80	74
	Range	65.64-118.81	43-86
$p=3,\ m=9,\ K=.25,\ n=35$	Median	165.96	67
	Range	88.22-174.47	50-110

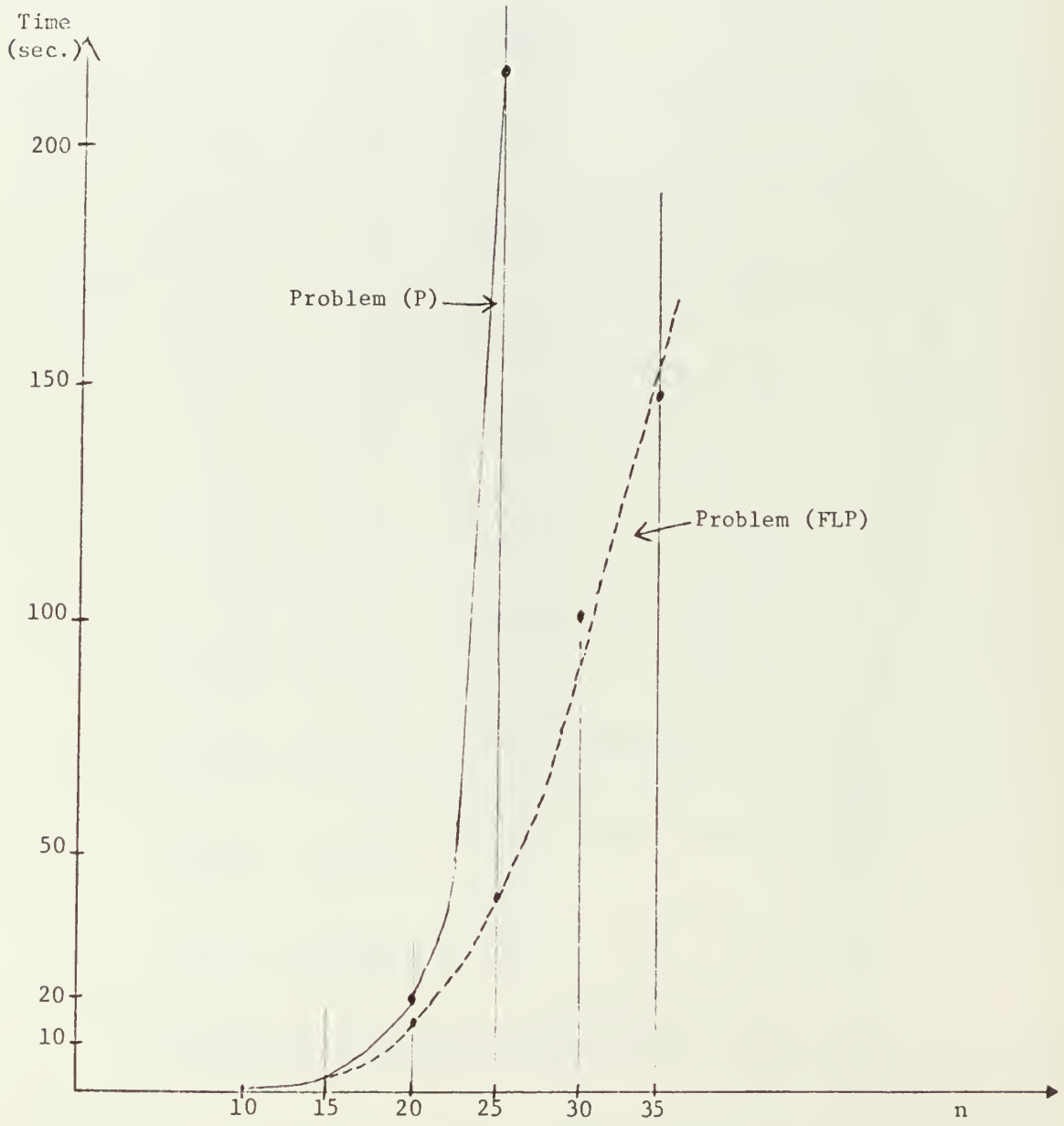

FIGURE 6. Comparison of results for (*P*) and (FLP)

3.3 Combining the Efficient Point and Utility Theory Approaches

The following utility function for each criterion i was considered for the purpose of illustration of the procedure.

$$u_i(x) = a_i(c_i - e^{-b_i x}).$$

The utility function of the decision maker was assumed to be additive, i.e.,

$$u = k_1u_1 + k_2u_2 + k_3u_3 \text{ with } \sum_{i=1}^{3} k_i = 1.$$

The parameters were chosen as follows. k_1, k_2, and b_i were generated randomly, while a_i and c_i were determined such that the maximum and minimum values of $u_i(\cdot)$ be, respectively, one and zero. In a real life application the parameters b_i, k_1, and k_2 must be assessed using, for example, the procedures in [4]. Other forms of utility function would not alter the methodology described here as long as they are monotonic nondecreasing in each criterion.

As mentioned earlier, the method we propose consists in first determining the set of efficient points and next performing an extensive sensitivity analysis, of the assessed parameters, of the utility function of the decision maker over the set *EF*. We recall that under the mild condition that the utility function be monotonic nondecreasing in each objective, it will attain its maximum over the feasible set at least one efficient point.

In our example, the sensitivity analysis was performed by varying the selected coefficients in a $\pm 10\%$ interval around their original values. The results are reported in a matrix form when only two parameters are varied. Table 5 illustrates such a situation. The parameters k_1 and k_2 were varied incrementally by .01 k_i over the intervals [.90 k_1, 1.10 k_1] and [.90 k_2, 1.10 k_2] and the resulting utility function value was computed for all points in *EF*. The code corresponding to the point that maximizes the utility of the decision maker is given in the matrix. We have indicated in the table the regions where a given efficient point remains optimal.

When more than two parameters are varied at the same time, the matrix representation is not applicable. In this case, the efficient points that maximize the utility function for some combination of the parameters are indicated together with the percentage of cases tested where they were the preferred choice of the decision maker. In Table 6 we have given an example of this procedure. Five parameters were varied, b_1, b_2, b_3, k_1, and k_2. Random numbers in the intervals [.9 b_i, 1.10 b_i] and [.9 k_j, 1.10 k_j], $i = 1, 2, 3$ and $j = 1, 2$ were generated and the resulting utility function was computed for all elements in *EF*. The percent of cases in which each efficient point is optimal is indicated in the table. In this case, for example, efficient point number 9 had the highest value of the utility function is sixty nine percent of the cases.

Both problems in Tables 5 and 6 are of the facility location type. Tables 5 and 6 suggest that the number of alternatives to be presented to the decision maker when the two approaches are combined is likely to be manageable. In this way, the analyst does not have the unpleasant task of presenting the entire set of efficient points to the decision maker and at the same time he can draw on the strengths of utility theory.

TABLE 5: *Utility Function Sensitivity Analysis Varying k_1 and k_2*

p	n	m	K
3	25	6	0.25

UTILITY FUNCTION COEFFICIENTS

k_1	k_2
0.60	0.19

UTILITY FUNCTION SENSITIVITY ANALYSIS

	$0.9k_1$								k_1											$1.1k_1$
$0.9k_2$	42	42	42	42	42	42	42	42	46	46	46	46	46	46	46	46	46	46	46	46
	42	42	42	42	42	42	42	42	46	46	46	46	46	46	46	46	46	46	46	46
	42	42	42	42	42	42	42	46	46	46	46	46	46	46	46	46	46	46	46	46
	42	42	42	42	42	42	46	46	46	46	46	46	46	46	46	46	46	46	46	48
	42	42	42	42	42	46	46	46	46	46	46	46	46	46	46	46	46	46	46	48
	42	42	42	42	42	46	46	46	46	46	46	46	46	46	46	46	46	46	48	48
	42	42	42	42	46	46	46	46	46	46	46	46	46	46	46	46	46	46	48	48
	42	42	42	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	48	48
	42	42	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	48	48	48
	42	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	48	48	48
k_2	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	48	48	48	48
	46	46	46	46	46	46	46	46	46	46	46	46	46	46	46	48	48	48	48	48
	46	46	46	46	46	46	46	46	46	46	46	46	46	46	48	48	48	48	48	48
	46	46	46	46	46	46	46	46	46	46	46	46	46	46	48	48	48	48	48	48
	46	46	46	46	46	46	46	46	46	46	46	46	46	48	48	48	48	48	48	48
	46	46	46	46	46	46	46	46	46	46	46	46	48	48	48	48	48	48	48	48
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	46	46	46	46	46	46	46	46	46	48	48	48	48	48	48	48	48	48	48	48
	46	46	46	46	46	46	46	46	48	48	48	48	48	48	48	48	48	48	48	48
$1.1k_2$	46	46	46	46	46	46	46	48	48	48	48	48	48	48	48	48	48	48	48	48

****ST
BOTTOM

TABLE 6 — *Output of the Algorithm*

p	n	m	K
3	35	6	0.25

Q	NFA	NTB
377	824	2867

EFFICIENT POINTS

1	430.51	307.94	334.08
2	354.14	426.28	387.93
3	425.10	416.42	274.17
4	426.28	387.93	392.48
5	437.69	328.42	373.54
6	270.06	320.49	428.53
7	253.67	271.97	437.69
8	429.71	329.32	406.60
9	434.27	376.52	402.53
10	307.12	434.27	290.54
11	431.09	395.85	382.81
12	427.67	351.24	406.72
13	271.97	437.69	328.42
14	428.16	203.54	413.49
15	353.62	431.09	395.85
16	391.50	419.10	321.10
17	413.22	417.93	413.22
18	355.64	425.10	416.42
19	417.93	413.22	427.67
20	384.50	307.12	434.27

UTILITY FUNCTIONS

$$u = k_1u_1 + k_2u_2 + k_3u_3 \qquad u_i = a_i(c_i - e^{-b_i x}) \qquad i = 1,2,3$$

$$\sum_{i=1}^{3} k_i = 1$$

COEFFICIENTS OF UTILITY FUNCTION

a_1	a_2	a_3	c_1	c_2	c_3
5.903	3.637	7.386	0.668	0.517	0.696

b_1	b_2	b_3	k_1	k_2
-0.002	-0.003	-0.001	0.600	0.192

COEFFICIENTS b_1, b_2, b_3, k_1 AND k_2 ARE RANDOMLY SELECTED BETWEEN + AND - 10% OF THE VALUES ABOVE, FOR PURPOSE OF SENSITIVITY.

UTILITY FUNCTION SENSITIVITY ANALISIS:
PERCENT OF CASES EACH EFFICIENT POINT IS OPTIMAL.

EFFICIENT N-:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
FREQUENCY :	0	0	4	0	0	0	0	0	69	0	17	0	0	0	0	0	0	0	10	0

4. COMPARISONS WITH EXISTING ALGORITHMS, AND CONCLUSIONS

A direct comparison of the c.p.u. times obtained with other results available in the literature is not straightforward, mainly because the corresponding experiments were undertaken on different computers. However, we report for the first time results on the number of efficient points and c.p.u. times for problems with up to 25 variables in the general structure problem (P) and up to 35 variables in the special structure problem (FLP) with 8 constraints.

Table 7 contrasts our results with those of Villarreal and Karwan [11] and Bitran [2].

TABLE 7 - *Comparison of Solution Times (c.p.u. Seconds) for Problems (P) of Size $m = 4$, $n = 10$, and $p = 3$*

K	Implicit Enumeration*	Villarreal and Karwan [11]**	Bitran [2]***
0.75	problem 1 1.39 problem 2 0.78	5.55	0.63
0.50	problem 1 1.13 problem 2 0.73	2.64	1.50
0.25	problem 1 0.30 problem 2 0.17	0.49	1.11

*Largest and smallest solution times in a sample of 5 problems using a computer PRIME 400.

**Smallest solution time reported by Villarreal and karwan with a computer CDC 6400.

***Smallest solution time reported by Bitran on a computer Burroughs B6700.

The combination of the efficient point and utility theory approaches that we have explored draws on the strengths of both methods. The computational results conducted suggest that the methodology developed reduces the task of the decision maker to choosing among a small number of alternatives. The results illustrated in Tables 5 and 6 are typical of those obtained in a more extensive computational analysis that we have performed.

The lower computation times observed for problems of type (FLP) suggest that there might be much to gain by developing methods for special structures. Another topic to be explored is the comparison in practical settings of the approach proposed in this paper with the Filtering Method introduced by Steuer [9]. The purpose of the Filtering Method is to avoid presenting the whole efficient set to the decision maker by selecting one representative of each cluster.

ACKNOWLEGMENT

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THE DISCRETE SEARCH PROBLEM AND THE CONSTRUCTION OF OPTIMAL ALLOCATIONS

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ABSTRACT

Suppose one object is hidden in the k-th of n boxes with probability $p(k)$. The boxes are to be searched sequentially. Associated with the j-th search of box k is a cost $c(j,k)$ and a conditional probability $q(j,k)$ that the first $j-1$ searches of box k are unsuccessful while the j-th search is successful given that the object is hidden in box k.

The problem is to maximize the probability that we find the object if we are not allowed to offer more than L for the search. We prove the existence of an optimal allocation of the search effort L and state an algorithm for the construction of an optimal allocation. Finally, we discuss some problems concerning the complexity of our problem.

1. INTRODUCTION

An oblivious person loses his glasses again and again in his own house which consists of n rooms. He has to decide how he will search for the glasses. Moreover, he has to notice that he may overlook the glasses when he searches the right room. In order to search "optimally" he estimates the following data: $p(k)$ for $k \in B = \{1, \ldots, n\}$, the a priori probability that he has lost the glasses in the k-th room, $c(j,k) \in \mathbb{R}^+$ for $j \in \mathbb{N}$ and $k \in B$, the length or cost of the j-th search of room k and $q(j,k)$ for $j \in \mathbb{N}$ and $k \in B$, the conditional probability that the first $j-1$ searches of room k are unsuccessful while the j-th search of room k is successful given that the glasses have been lost in room k. Since these events exclude each other $\sum_{1 \leqslant j < \infty} q(j,k) \leqslant 1$.

The length of an inspection surely depends on the searched room but it is also reasonable to assume that it depends on the number of previous searches. On the one hand some parts of the room are clearly arranged so that we have to search them only once, on the other side we open the cupboards not before the fourth search of a room.

More important situations where we can apply this search problem are the following: the search for a defect in a large system, the search for oil or a mineral in different regions and the search for a sunken ship where we partition the ocean to map squares. In the following we shall talk about the search for an object hidden in one of n boxes.

We have to decide what we intend by an optimal search. We may try to minimize the expected cost of a successful search. This problem has been solved by the author [6]. But often we are not able to search for the hidden object until it is found. If we want to visit the

theater the time for the search of our glasses is limited. Looking for a sunken ship we may hope only for a time period of length T that we save survivors. A company digging for a mineral has only limited funds. The best we can do in these situations is to maximize the probability of a successful search without offering more than a given limit $L \in \mathbb{R}_0^+$.

For this problem we do not have to decide in which order we will search the boxes but only how often we want to search each box. Each function $a : B \to \overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$ is an allocation: search box k for $a(k)$ times ($k \in B$). If we use an allocation a we have to be able to pay all the prescribed searches. The cost of these searches amounts to $C(a) := \sum_{k \in B} \sum_{1 \leqslant j \leqslant a(k)} c(j,k)$ ($c(\infty,k) = 0$). Because we shall consider only allocations whose cost does not exceed the given limit L we call an allocation a L-admissible iff $C(a) \leqslant L$. We find the object during the j-th search of box k iff it is hidden there; we have overlooked it for $j-1$ times and do not overlook it during the j-th search of k. The probability of this event is obviously $p(k)q(j,k)$. Therefore $P(a)$, the probability of the success of our allocation, comes to

$$P(a) := \sum_{k \in B} \sum_{1 \leqslant j \leqslant a(k)} p(k)q(j,k) \quad (q(\infty,k) = 0).$$

a is L-optimal if it is L-admissible and no other L-admissible allocation has a larger probability of success. If we have computed an L-optimal allocation a we may use the results of [6] to decide in which order we should realize the searches which are prescribed by a.

Kadane [3] has investigated this problem. He assumes that $e(j,k) := p(k)q(j,k)c(j,k)^{-1}$ is decreasing for each k as a function of j and states an algorithm which computes an L-optimal allocation if it stops. Kadane notes that his algorithm stops if c is bounded below by a constant $\epsilon > 0$. This is not surprising because in that case the number of L-admissible allocations is finite. We give a generalization of Kadane's algorithm and we prove that it stops for each search problem and that it computes always an L-optimal allocation. Arkin [2] and Stone [5] deal with even more special cases than Kadane.

We shall now discuss why the assumption of Kadane simplifies the problem. $p(k)q(j,k)$ is the probability that the search (j,k), i.e., the j-th search of box k, is successful while the cost of this search amounts to $c(j,k)$. The quotient of the profit and the cost, $e(j,k)$ is the efficiency of this search. Now it seems to be reasonable that we try to perform the most efficient searches. If the assumption of Kadane is not fulfilled the most efficient search may be the seventh search of box k. But we cannot perform this search without realizing the first six searches of box k which may be very inefficient.

2. THE DUAL SEARCH PROBLEM

First we introduce some definitions which will simplify the notation. $p(j,k) := p(k)q(j,k)$ is the probability of success of the search (j,k). $p^*(j,k) := \sum_{1 \leqslant i \leqslant j} p(i,k)$ is the probability of success and $c^*(j,k) := \sum_{1 \leqslant i \leqslant j} c(i,k)$ is the cost of the first j searches of box k. For each allocation a we conclude $P(a) = \sum_{k \in B} p^*(a(k), k)$ and $C(a) = \sum_{k \in B} c^*(a(k), k)$.

For each box k let f_k be the following step map on $[0, c^*(\infty,k))$. For $x \in [c^*(j-1,k), c^*(j,k))$ let $f_k(x) := p^*(j-1,k)$ and $f_k(c^*(\infty,k)) := \lim_{x \to c^*(\infty,k)} f_k(x)$.

Then $f_k(c^*(j,k)) = p^*(j,k)$ and $P(a) = \sum_{k \in B} f_k(c^*(a(k), k))$. Let g_k be the continuous function which combines linearly the points $(c^*(j,k), p^*(j,k))$ and let h_k be the smallest concave function nowhere smaller than f_k. Finally, let $B(k) := \{j \in \bar{\mathbb{N}}_0 \mid f_k(c^*(j,k)) = h_k(c^*(j,k))\}$. Figure 1 illustrates these definitions.

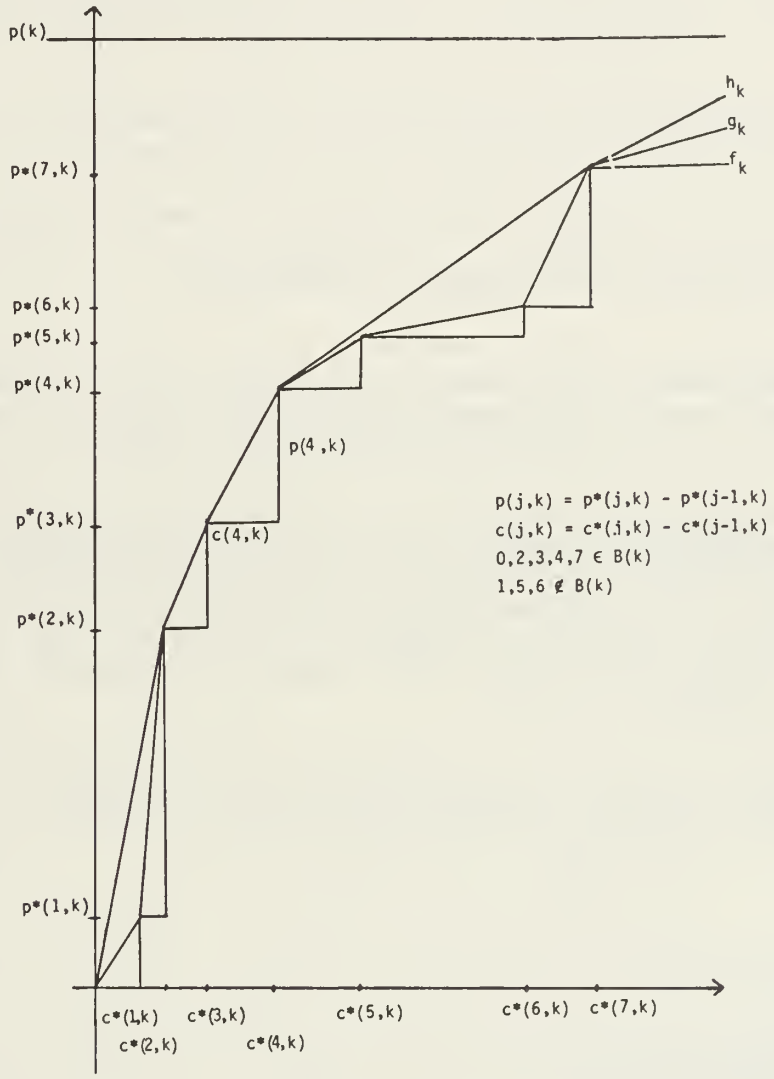

FIGURE 1. Efficiency of search

Obviously, $0, \infty \in B(k)$. The slope of g_k in an interval $(c^*(j-1, k), c^*(j,k))$ is equal to the efficiency of the search (j,k). If and only if the assumption of Kadane is fulfilled g_k is concave. This is equivalent to $g_k = h_k$ and $B(k) = \bar{\mathbb{N}}_0$.

In [6] it turned out that the searches $J+1, \ldots, J'$ of k "belong together" if J' is the direct successor of J in $B(k)$. In that paper the dual search problem to a search problem is defined by combining the searches which belong together. The dual problem is useful for this paper too.

DEFINITION 1: The dual search problem of a search problem is defined by the parameters n', p', c', q'. $n' := n$, $p' := p$. If $B(k)$ has infinitely many elements, we enumerate $B(k) - \{\infty\}$ in the following way: $0 =: B_{0,k} < B_{1,k} < \ldots$. Let $c'(j,k) := \sum_{B_{j-1,k} < i \leqslant B_{j,k}} c(i,k)$ and $q'(j,k) := \sum_{B_{j-1,k} < i \leqslant B_{j,k}} q(i,k)$. If $B(k)$ has finitely many elements we enumerate $B(k)$ by $0 =: B_{0,k} < \ldots < B_{m(k),k} := \infty$. For $j \leqslant m(k)$ we define $c'(j,k)$ and $q'(j,k)$ as before, and for $j > m(k)$ we define additional searches by $c'(j,k) := 1$ and $q'(j,k) := 0$.

Looking at Figure 1 it is easy to see that $g'_k = h_k$. Therefore, g'_k is concave and the assumption of Kadane is always fulfilled for the dual search problem.

Before we try to compute an L-optimal allocation we prove its existence.

3. THE EXISTENCE OF L-OPTIMAL ALLOCATIONS

Since the set of L-admissible allocations is never empty (let $a(k) = 0$ for all k) and since the probability of success of an allocation never exceeds 1 $P_{\sup} := \sup\{P(a) | a \text{ } L\text{-admissible}\}$ is always well defined. There exists always an L-admissible allocation a_i^0 so that $P(a_i^0) \geqslant P_{\sup} - 2^{-i}$. We note that we don't know how to compute a_i^0.

In order to prove the existence of an L-optimal allocation we define a subsequence a_i^n of a_i^0 with some additional properties. If $a_i^{k-1} (1 \leqslant k \leqslant n)$ is already defined, we define a_i^k as a subsequence of a_i^{k-1} in the following way. If $a_i^{k-1}(k)$ is bounded we choose a_i^k so that $a_i^k(k)$ is constant for $i \in \mathbb{N}$. Otherwise, we choose a_i^k so that $a_i^k(k) \geqslant i$ for $i \in \mathbb{N}$. By these definitions a_i^n is L-admissible and $P(a_i^n) \geqslant P_{\sup} - 2^{-i}$. If $a_i^n(k)$ $(i \in \mathbb{N})$ is a constant sequence let $a(k)$ be equal to this constant. The set of these boxes is called B'. Otherwise, $a_i^n(k) \geqslant i$. Let $a(k) := \infty$ and $B'' := B - B'$.

Because $P(a) \geqslant P(a_i^n) \geqslant P_{\sup} - 2^{-i}$ for all i we conclude that $P(a) \geqslant P_{\sup}$. In order to prove that a is L-optimal, it is sufficient to prove that a is L-admissible.

$$C(a) = \sum_{k \in B'} c^*(a(k), k) + \sum_{k \in B''} c^*(\infty, k)$$

$$= \lim_{i \to \infty} \left(\sum_{k \in B'} c^*(a(k), k) + \sum_{k \in B''} c^*(i,k) \right) \leqslant \overline{\lim_{i \to \infty}} C(a_i^n) \leqslant L.$$

Now we have proved

THEOREM 1: The set of L-optimal allocations is not empty. $P_{\max} := \max\{P(a) | a \text{ } L\text{-admissible}\}$ is well defined.

4. BOUNDS ON $P_{\max}$

First we give a lower bound on $P_{\max}$ by selecting an L-admissible allocation which seems to be rather good. Since for the dual search problem the assumption of Kadane that $e'(j,k)$ is for each k decreasing is fulfilled, we may fix a complete order on the set of all searches in such a way that (j,k) precedes (j',k') if $e'(j,k) \geqslant e'(j',k')$ and that $(j-1, k)$ always precedes (j,k). We note that in this order each search has a well-defined direct successor (but not always a direct predecessor).

By $a'(j,k)$ we denote that allocation which prescribes exactly all searches preceding (j,k) (but not (j,k)). (j,k) is called the crucial search of $a'(j,k)$. It is easy to prove (see [3] or [5]) that $a'(j,k)$ is $C'(a'(j,k))$-optimal. But we cannot choose how much we will offer. Our funds are limited by L. If the cost of all searches does not exceed L let $a' \equiv \infty$ be our L-admissible allocation which of course is L-optimal. Otherwise, there is exactly one search (j,k) so that $a' = a'(j,k)$ is L-admissible but $a'(j',k')$ where (j',k') is the direct successor of (j,k) is not L-admissible. a' seems to be a good L-admissible allocation for the dual problem.

In order to imitate a' by an L-admissible allocation we use the enumeration of $B(k)$ or $B(k) - \{\infty\}$ of Definition 1. If $a'(k) = \infty$ or if $a'(k) \geqslant m(k)$ (if $B(k)$ has finitely many elements) let $a(k) := \infty$. Otherwise, $a(k) := B_{a'(k),k}$. We have decomposed a search of the dual problem to its components from which it has been combined. We only omit the additional searches. Therefore, $C(a) \leqslant C'(a') \leqslant L$. We call a the good allocation for our search problem. If $a' = a'(j,k)$ and $j \leqslant m(k)$ we call the last of the components of the crucial search (j,k) for the dual problem the crucial search of a. That means $(B_{j,k},k)$ is the crucial search of a is $a' = a'(j,k)$ and $j \leqslant m(k)$.

PROPOSITION 1: The good allocation a is for each search problem L-admissible and $P(a) \leqslant P_{\max}$. We like to mention that Kadane has already used the allocation a'. For the computation of a good upper bound on $P_{\max}$, we introduce a new concept.

DEFINITION 2: A randomized allocation is a pair (a,π) where a is an allocation and $\pi : B \to [0,1]$ so that $\pi(k) = 0$ if $a(k) = \infty$. (a,π) prescribes the following procedure: search box k for a $a(k)$ times and then perform the next search of box k with probability $\pi(k)$. The expected cost of (a,π) is $C(a,\pi) := \sum_{k \in B} (c^*(a(k), k) + \pi(k)\, c(a(k) + 1, k))$ and its expected probability of success comes to $P(a,\pi) := \sum_{k \in B} (p^*(a(k), k) + \pi(k) p(a(k) + 1, k))$.

If $\pi \equiv 0$ we obtain a standard allocation. Let $\tilde{p}(a,\pi,k) := p^*(a(k), k) + \pi(k) p(a(k) + 1, k)$ and $\tilde{c}(a,\pi,k) := c^*(a(k), k) + \pi(k) c(a(k) + 1, k)$. Then we conclude by the definition that $\tilde{p}(a,\pi,k) = g_k(\tilde{c}(a,\pi,k))$ (see Figure 1) and $P(a,\pi) = \sum_{k \in B} g_k(\tilde{c}(a,\pi,k))$.

If the assumption of Kadane is fulfilled it is easy to define an L-optimal randomized allocation (a,π). Let a be the good allocation. If $a \equiv \infty$ let $\pi \equiv 0$ and (a,π) is obviously L-optimal. Otherwise, $a = a(j,k)$. Let $\pi(k') = 0$ for $k' \neq k$ and $\pi(k) := (L - C(a)) c(j,k)^{-1}$. Obviously, $C(a,\pi) = L$. That (a,π) is L-optimal may be proved by an application of the generalized Neyman-Pearson-lemma (see Kadane [3]): (a,π) prescribes all searches whose efficiency is larger than $e(j,k)$, no search whose efficiency is smaller than $e(j,k)$ and as many of the searches with efficiency $e(j,k)$ that $C(a,\pi) = L$.

PROPOSITION 2: Let (a', π') be an L-optimal allocation for the dual search problem. Then $P_{\max} \leqslant P'(a', \pi')$.

PROOF: Since the assumption of Kadane is fulfilled for the dual problem, we can compute (a', π') by the procedure above.

Let a be an L-optimal allocation for the original problem which exists by Theorem 1. Then $P_{\max} = P(a) = \sum_{k \in B} f_k(c^*(a(k), k)) \leqslant \sum_{k \in B} h_k(c^*(a(k), k)) = \sum_{k \in B} g'_k(c^*(a(k), k))$. The last equality follows from the fact that the function h_k for the original problem equals the function g'_k for the dual problem (see Section 2).

Now we imitate a by an L-admissible randomized allocation $(\bar{a}, \bar{\pi})$ for the dual problem. We use the enumeration of $B(k)$ or $B(k) - \{\infty\}$ of Definition 1. If $a(k) = \infty$ let $\bar{a}(k) := \infty$ if $B(k)$ has infinitely many elements and $\bar{a}(k) := m(k)$ otherwise. $\bar{\pi}(k) := 0$ in both cases. Otherwise, there exists $l(k)$ so that $B_{l(k),k} \leqslant a(k) < B_{l(k)+1,k}$. Let $\bar{a}(k) := l(k)$ and $\bar{\pi}(k) := (c^*(a(k), k) - c^*(\bar{a}(k) \quad k)) \, c'(\bar{a}(k) + 1, k)^{-1}$. It follows $\tilde{c}'(\bar{a}, \bar{\pi}, k) = c^*(\bar{a}(k), k) + \bar{\pi}(k) \quad c'(\bar{a}(k) + 1, k) = c^*(a(k), k)$ and $C'(\bar{a}, \bar{\pi}) = C(a) \leqslant L$. Thus, $(\bar{a}, \bar{\pi})$ is L-admissible. Finally, we combine our results

$$P_{\max} \leqslant \sum_{k \in B} g'_k(c^*(a(k), k)) = \sum_{k \in B} g'_k(\tilde{c}'(\bar{a}, \bar{\pi}, k)) = P'(\bar{a}, \bar{\pi})$$
$$\leqslant P'(a', \pi'). \qquad \text{Q.E.D.}$$

5. THE DECOMPOSITION OF A SEARCH PROBLEM

We are now able to compute a lower and an upper bound on $P_{\max}$. If these bounds coincide the good allocation a of proposition 1 is L-optimal. Otherwise we don't know whether a is L-optimal. Analogously to [3] we divide the problem to two subproblems. Let $A(L)$ be the set of all L-admissible allocations,

$$A'(L) := \{a \in A(L) | a(k) < j\} \text{ and } A''(L) := \{a \in A(L) | a(k) \geqslant j\}.$$

The computation of an optimal allocation for the first (second) subproblem is equivalent to the computation of a best of all allocations of $A'(L)$ $(A''(L))$. We "exclude" ("include") the search (j, k).

For both problems we compute lower and upper bounds. If for the undivided problem with the largest upper bound the lower and the upper bound coincide, we computed an L-optimal allocation. Otherwise, we divide this problem and so on. Since we don't suppose that the assumption of Kadane is fulfilled, we have to choose more carefully than Kadane the search for which we divide the problem. We have to ensure that the algorithm stops and we want to get an algorithm which is as fast as possible. At first we show how we divide a problem.

(i) The exclusion of the search (j, k).

Let $S' = (n', p', c', q', L')$ be the following search problem: $n' = n$, $p' = p$, $L' = L$, the searches (j', k') where $k' \neq k$ or $k' = k$ and $j' < j$ have the same cost and the same probability of success as before and for $j' \geqslant j$ let $c'(j', k) = L + 1$ and $q'(j', k) = 0$. Obviously, an allocation a is L'-admissible for S' iff it is an allocation of $A'(L)$ for the original problem S. In that case the probability of success of a is the same whether we use a for S or for S'. Thereupon, a is L'-optimal for S' iff it is the best of all allocations of $A'(L)$ for S.

(ii) The inclusion of the search (j, k).

Let $S'' = (n'', p'', c'', q'', L'')$ be the following search problem: $n'' = n$, $p'' = p$, $L'' = L - c^*(j, k)$, the searches (j', k') where $k' \neq k$ have the same cost and the same probability of success as before and for $i \in \mathbb{N}$ let $c''(i, k) = c(i + j, k)$ and $q''(i, k) = q(i + j, k)$.

$L'' < 0$ iff $L < c^*(j, k)$, i.e., iff the cost of the included searches is larger than our funds. Therefore, $L'' < 0$ iff $A''(L) = \emptyset$. For an L''-admissible allocation a'' for S'', we define an allocation $a \in A''(L)$ for S by $a(k') := a''(k')$ for $k' \neq k$ and $a(k) := a''(k) + j$. By this definition $C(a) - C''(a'') = c^*(j, k)$ and $a \in A''(L)$ for S iff a'' is L''-admissible for S''. Therefore, we have a one-to-one correspondence between $A''(L)$ for S and the set of L''-admissible allocations for S''. Since $P(a) - P''(a'') = p^*(j, k)$ a is the best of all allocations of $A''(L)$ for S iff a'' is L''-optimal for S''.

We show how we obtain an order on the searches of the dual problem of S''. For the included search (j, k), we assume $j \in B(k)$. Therefore, $j = B_{l,k}$ for some l (see Definition 1). If we erase the first l searches of k in the order of the searches of the dual problem of S and if we replace the search (i, k) for $i > l$ by the search $(i - l, k)$, we obtain an order on the searches of the dual problem of S''.

6. AN ALGORITHM FOR THE CONSTRUCTION OF AN L-OPTIMAL ALLOCATION

We have the task to compute an L_0-optimal allocation for the search problem $S_0 = (n_0, p_0, c_0, q_0, L_0)$, i.e., an allocation of $A_0(L_0)$ with the largest possible probability of success. In the course of our algorithm we partition $A_0(L_0)$ to disjoint subsets $A_i(L_0)$ by the inclusion and/or exclusion of some searches which will be determined later. Therefore, (see Section 5) the computation of a best of all allocations of $A_i(L_0)$ is equivalent to the solution of a search problem $S_i = (n_i, p_i, c_i, q_i, L_i)$.

We are able to compute functions $l_i, u_i : B \to \bar{\mathbb{N}}_0$ so that $A_i(L_0) = \{a \in A_0(L_0) \mid l_i(k) \leqslant a(k) \leqslant u_i(k)\}$. Let $l_0 \equiv 0$, $u_0 \equiv \infty$. If S_i is generated from S_t by the inclusion (exclusion) of the search (j, k), we conclude that $u_i \equiv u_t$, $l_i(k') = l_t(k')$ for $k' \neq k$ and $l_i(k) = l_t(k) + j$, resp. $l_i \equiv l_t$, $u_i(k') = u_t(k')$ for $k' \neq k$ and $u_i(k) = l_t(k) + j - 1$.

By I_i we denote the set of the indices of undivided search problems. For S_i we compute the good L_i-admissible allocation a^*_i of Proposition 1. By the results of the previous section $a_i := a^*_i + l_i$ is an L_0-admissible allocation for S_0 and $P_0(a_i) = P_i(a^*_i) + P_0(l_i)$ is a lower bound on $P^0_{\max}$. Furthermore, a_i is the best of all allocations of $A_i(L_0)$ iff a^*_i is L_i-optimal for S_i. Then we compute by Proposition 2 an upper bound U^*_i on $P^i_{\max}$, the probability of success of an L_i-optimal allocation for S_i. Again by the results of the previous section we can conclude that $U_i := U^*_i + P_0(l_i) \geqslant \max\{P_0(a) \mid a \in A_i(L_0)\}$. Since the sets $A_m(L_0)$ $(m \in I_i)$ form a partition of $A_0(L_0)$ $\max\{U_m \mid m \in I_i\}$ is an upper bound on $P^0_{\max}$.

Finally, we have to discuss how we divide the search problem S_t with the maximal value of $U_m (m \in I_i)$. If $P_0(a_t) = U_t$ the allocation a_t is L_0-optimal for S_0 since always $P_0(a_t) \leqslant P^0_{\max} \leqslant U_t$. If a^*_t prescribes for S_t all searches with positive efficiency $P_t(a^*_t) = U^*_t$ and therefore, $P_0(a_t) = P_t(a^*_t) + P_0(l_t) = U^*_t + P_0(l_t) = U_t$.

Otherwise, we have defined in Section 4 a crucial search (j_t, k_t) for a^*_t where $j_t \in B_t(k_t)$. If $j_t < \infty$ we divide S_t by the inclusion (exclusion) of this crucial search. We try to give reasons for this choice. If the assumption of Kadane is fulfilled and if we realize the searches according to their fixed order, the crucial search is the first search which we are not able to realize since our funds are exhausted. This justifies the name *crucial search.* In order to obtain a fast algorithm it seems to be reasonable to choose this search for the decomposition of S_t. In general case we imitate the above procedure by imitating the above allocation for the dual problem. Let (j'_t, k'_t) be the first search which we are not able to realize since our funds are

exhausted. Then $k_t = k'_t$ and $j_t = \min \{i \in \bar{\mathbb{N}} \mid i \geqslant j'_t, i \in B_t(k_t)\}$. We again divide the problem by the search (j_t, k_t) (and not by (j'_t, k'_t)) because this ensures that the algorithm stops.

If $j_t = \infty$ it makes no sense to exclude (j_t, k_t). If we decide to divide S_t by (j'_t, k'_t) the algorithm may not stop. In order to ensure that the algorithm stops we divide S_t to more than two subproblems. In doing so we guarantee that for the search problem where no search has been excluded the funds do not exceed 0. Let us imitate that allocation for the dual problem which prescribes in addition to the searches of the good allocation the crucial search. The result is b_t where $b_t(k) = a_t(k)$ for $k \neq k_t$ and $b_t(k_t) = \infty$. Since $C_t(b_t) > L_t$ we may choose an allocation $b'_t : B \to \mathbb{N}_0$ so that $b'_t(k) \leqslant b_t(k)$ and $C_t(b'_t) > L_t$. Let $\{k^1, \ldots, k^r\} := \{k \in B \mid b'_t(k) > 0\}$ and $R := 2^r$. We divide S_t to R subproblems which correspond to the R different sequences "inclusion/exclusion of $(b'_t(k^1), k^1)$, ..., inclusion/exclusion of $(b'_t(k^r), k^r)$." Since $b'_t(k^i) < \infty$ these inclusions and exclusions are well defined. Finally, we use the convention that $U_i := -1$ iff $L_i < 0$, i.e., iff $A_i(L_0) = \emptyset$.

For the search problem S_i the computation of the necessary data means the computation of a_i, $P(a_i)$, U_i and $-$ if defined $-$ (j_i, k_i) if $L_i \geqslant 0$ and the definition $U_i := -1$ if $L_i < 0$.

THE ALGORITHM

Input: the search problem $S_0 = (n_0, p_0, c_0, q_0, L_0)$.

Step 1: $L_0 < 0 \Longrightarrow A_0(L_0) = \emptyset$ → STOP
$L_0 \geqslant 0 \Longrightarrow i := 0$, $I_0 := \{0\}$, compute the necessary data for S_0 → Step 2

Step 2: Choose $t \in I_i$ so that $U_t = \max \{U_m \mid m \in I_i\}$.
$P(a_t) = U_t \Longrightarrow a_t$ is L_0-optimal → STOP
$P(a_t) \neq U_t$, $j_t < \infty$ → Step 3
$P(a_t) \neq U_t$, $j_t = \infty$ → Step 4

Step 3: Divide S_t by the inclusion/exclusion of (j_t, k_t) to S_{i+1} and S_{i+2}. Compute the necessary data for S_{i+1} and S_{i+2}. Increase i by 2. Set $I_i = (I_{i-2} \cup \{i-1, i\}) - \{t\}$ → Step 2

Step 4: Divide S_t by the R sequences inclusion/exclusion of $(b'_t(k^1), k^1)$, ..., inclusion/exclusion of $(b'_t(k^r), k^r)$ to $S_{i+1}, \ldots, S_{i+R}$. Compute the necessary data for $S_{i+1}, \ldots, S_{i+R}$. Increase i by R. Set $I_i = (I_{i-R} \cup \{i-R+1, \ldots, i\}) - \{t\}$ → Step 2

Combining our results we have proved

THEOREM 2: If $L_0 \geqslant 0$ and the algorithm stops a_t is an L_0-optimal allocation.

7. THE ANALYSIS OF THE ALGORITHM

THEOREM 3: The algorithm stops for each search problem.

PROOF: We define for each search problem a characteristic (V, W). V is the number of boxes for which infinitely many searches have positive efficiency. For the other boxes let the $W(k)$-th search be the last search with positive efficiency. W is the sum of these $W(k)$. We prove the theorem by induction on the lexicographical order on $\mathbb{N}_0^2$: (V', W') is smaller than (V, W) iff $V' < V$ or $V' = V$ and $W' < W$.

For $(V, W) = (0, 0)$ all searches have efficiency 0 and the algorithm stops by the first passage of Step 1 or 2. Now let the assertion of the theorem be valid for all (V', W') smaller than (V, W).

Let S be a search problem with the characteristic (V, W). If the algorithm stops by the first passage of Step 1 or 2 the assertion of the theorem is valid for S. Otherwise, S becomes divided. We compute an optimal allocation for S while we compute for all subproblems an optimal allocation by the algorithm and while we compare these allocations. This new procedure is at most as fast as the algorithm. Therefore, it is sufficient to prove the assertion for all subproblems S'.

If we include or exclude a search the characteristic (V', W') of the new problem is at most as large as the characteristic (V, W) of the given problem.

CASE 1: During the definition of S', we have excluded some search.

(i) If we have excluded a search (j,k) so that infinitely many searches of k had positive efficiency $V' < V$ and the algorithm stops for S' by the induction hypothesis.

(ii) If (i) is not fulfilled and if we have excluded a search (j,k) where $j \leqslant W(k)$, we conclude that $V' = V$, $W'(k) < W(k)$ and $W'(k') \leqslant W(k')$ for $k' \neq k$. Therefore, $W' < W$ and the algorithm stops for S' by the induction hypothesis.

(iii) If (i) and (ii) are not fulfilled we have excluded a search (j,k) where $j > W(k)$. This is impossible by the definition of our algorithm. If (j,k) is the crucial search (Step 3) the good allocation prescribes all searches with positive efficiency and the algorithm would have stopped at Step 2. If (j,k) is excluded at Step 4 $j \leqslant W(k)$ by the definition of b'.

CASE 2: S' has been defined at Step 4 and we have not excluded some search. The cost of the included searches is larger than our funds. Therefore, $L' < 0$ and the algorithm stops immediately.

CASE 3: S' has been defined at Step 3 and we have not excluded some search. If (V', W') is smaller than (V, W) the algorithm stops for S' by the induction hypothesis. Otherwise, $(V', W') = (V, W)$. Either the algorithm stops at Step 1 or 2 and the assertion is proved or S' becomes divided. We again argue that it is sufficient to prove the assertion for all subproblems S'' of S' and again we obtain at most one crucial problem. We repeat this argumentation for $n + 1$ times. If the theorem is not proved we always obtain a new crucial problem with the characteristic (V, W) by including a search at Step 3. Well call these problems $S_0 := S$, $S_1 := S'$, S_2, ..., S_{n+1}.

S_{i+1} is generated from S_i by the inclusion of (j_i, k_i) at Step 3 $(0 \leqslant i \leqslant n)$. While we show that this case never happens we prove the theorem.

Since $j_i \in B_i(k_i)$ the end of the search (j_i, k_i) coincides with the end of a search (j'_i, k_i) for the dual problem. We remember that we obtain the order on the searches of the dual problem of S_{i+1} by erasing the first j'_i searches of k_i in the order of the searches of the dual problem of S_i. Let a'_i be the allocation for the problem of S_i which prescribes all searches preceding (j'_i, k_i) and (j'_i, k_i), too.

We know by the definition of (j_i', k_i) that the cost of a_i' exceeds L_i. By the definition of the inclusion of (j_i, k_i) the cost of the searches which have preceded (j_i', k_i) for the dual problem of S_i and which we have not erased exceeds L_{i+1} for the dual problem of S_{i+1}. Among these searches is no search of box k_i. By the definition of the crucial search (j_{i+1}', k_{i+1}), we conclude that $a_{i+1}'(k) \leqslant a_i'(k)$ for all k and $a_{i+1}'(k_i) = 0$ while $a_i'(k_i) = j_i' > 0$. Thus, $k_0, \ldots, k_n$ are distinct which obviously is impossible. Q.E.D.

We finish this paper with some comments about the algorithm.

(1) If the assumption of Kadane is fulfilled we never reach Step 4 and our algorithm coincides with the algorithm of Kadane.

(2) At each time $\max \{P_0(a_m) | m \in I_i\} \leqslant P_{\max}^0 \leqslant \max \{U_m | m \in I_i\}$. In order to obtain only a nearly optimal L_0-admissible allocation, we may stop the algorithm if the difference of the two bounds is small enough.

(3) Always using Step 4 instead of Step 3, we obtain also an algorithm which always stops and which always computes an optimal allocation. The proof of Theorem 3 would be much easier since Case 3 becomes superfluous. But it seems to be obvious that our algorithm is much faster for nearly all search problems.

(4) Obviously our algorithm is not very fast. It is not even polynomially time bounded. Let us consider the following assumptions for our search problem: $c(2,k) = L + 1$, $c(1,k) \in \mathbb{N}$, $L \in \mathbb{N}$ and $p(1,k) = d\, c(1,k)$ where $d \in \mathbb{R}^+$. Then we may restrict ourselves to allocations $a: B \to \{0,1\}$. Such an allocation corresponds to the set $A := \{k | a(k) = 1\}$. We conclude $C(a) = \sum_{k \in A} c(1,k)$ and $P(a) = \sum_{k \in A} p(1,k) = d\, C(a)$. Therefore, a is L-optimal iff $C(a)$ is maximal under the restriction that $C(a) \leqslant L$. While computing an L-optimal allocation we decide whether there exists an allocation a where $C(a) = L$. This problem is known as the knapsack problem. It is well known that the knapsack problem is NP-complete (Karp [4]). If the conjecture of many computer scientists that $NP \neq P$ is correct there is not fast algorithm for the knapsack problem. In particular, there is no fast algorithm for our search problem. A more detailed discussion of these ideas may be found in [1].

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SEARCH TECHNIQUES FOR A NONLINEAR MULTIPROCESSOR SCHEDULING PROBLEM

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ABSTRACT

The problem of assigning computer program modules to functionally similar processors in a distributed computer network is investigated. The modules of a program must be assigned among processors in such a way as to minimize interprocessor communication while taking advantage of affinities of certain modules to particular processors. This problem is formulated as a zero-one quadratic programming problem, but is more conveniently modeled as a directed acyclic search graph. The model is developed and a backward shortest path labeling algorithm is given that produces an assignment of program modules to processors. A non-backtracking branch-and-bound algorithm is described that uses a local neighborhood search at each stage of the search graph.

1. INTRODUCTION

Developments in hardware technology and the trend toward large shared data base systems have contributed to the feasibility and popularity of distributed computing systems. A distributed computer network is considered to be a set of programmable processors interconnected to some extent by communication facilities [14]. Cost-effective methods must be developed for these systems to control the allocation of computing resources among the jobs introduced into the network.

Scheduling theory [6] is a subject that has received a great deal of attention since the development of digital computer systems. The general scheduling problem is that of allocating limited resources among multiple tasks when choices exist in the allocation process. The policies governing the apportionment of the resources are called scheduling rules or scheduling algorithms. Scheduling algorithms for computer systems are used to assign a set of jobs to resources which are used in executing or servicing the jobs [4, 5, 17].

The nature of job scheduling [24] depends on the functional similarity of the processing nodes and on the degree of communication available between processors. If the network consists of functionally different processors, then job scheduling is simple since each job would be designed for, and therefore assigned to, one particular specialized processor.

In a network of functionally similar processing nodes, it may be possible to assign the parts of a program freely among the processors; but in a practical sense, the communication links in a distributed network constitute inherent bottlenecks and therefore constrain the assignment of computational tasks. When high penalties are imposed for communication, the practical solution is to minimize the amount of communication between processors by assigning related tasks to the same processor. However, if the processors in the network were fully connected by high capacity data links, many feasible alternative assignments of computational tasks to processors would exist and should be evaluated by the job scheduler. In such cases, interprocessor communication would no longer be regarded as a serious constraint, and indeed the overall efficiency of the system would be improved through intelligent scheduling that takes such communication into account.

Recent technological advances, such as the economical fabrication of processors and the development of broadband communication facilities, suggest that this is precisely the character of distributed processing that will become prevalent in the near future [31]. In fact, the architecture of the *Cm** multimicroprocessor system [23] provides a good example of the type of distributed computers that will be considered in this paper; and the problem of distributing software *utilities* among *Cm** processors is similar to the assignment problem studied here.

In Section 2, the problem is stated and formulated as a nonlinear programming problem. In Section 3, a directed acyclic search graph is described and a shortest path algorithm is developed that yields an assignment. A branch-and-bound algorithm, described in Section 4, may be applied to the search graph to produce assignments of generally inferior quality but with considerably less computational effort. An analysis of the algorithms and their performance is given in the last section.

2. STATEMENT OF THE PROBLEM

The distributed system under discussion is assumed to be a network of processors, fully-connected by identical communication links (so that the cost of sending one unit of data between any two processors is the same). Although the processors are assumed to be functionally similar, they need not be identical. In fact, certain processors may have particular efficiencies for executing particular program modules. For example, some processors may have high-speed arithmetic capabilities, access to a needed data base, a large high-speed memory, availability to required peripheral devices, or other facilities associated with them which make them especially well-suited for executing certain program modules.

The programs being executed within the distributed computer system are assumed to be partitioned into functional modules (containing executable code and/or data) which, in general, may reside on any processor in the system. There is no parallelism or multitasking of module execution within a program. Each processor may be multiprogrammed, and divide its time among several programs, but concurrent execution of the modules in one program is not considered.

The modules of a modular program are to be assigned among the processors in such a way as to take advantage of the affinities of some modules to certain processors while minimizing the costs of communication between jobs that are assigned to different processors. Thus, there are two kinds of costs that must be considered in the search for an assignment:

a. Each module has an execution cost that depends on the processor to which it is assigned. Let e_{ij} represent the cost of executing module i on processor j.

b. Any two modules that communicate during program execution incur a penalty if they are assigned to different processors. (It is assumed that the cost of such communication is zero when the reference is made between modules residing on the same processor.) Let the cost of communication between program modules i and k be denoted by c_{ik}. The costs e_{ij} and c_{ik} must be measured in the same units of money or time.

The distributed programs considered here are serial programs for which execution can shift from one processor to another. Therefore, all execution costs and communication costs are incurred in disjoint time intervals, and if these costs are measured in time, then the cost of the assignment is actually the completion time of the program.

The desired assignment is one which minimizes the sum of the execution costs of the modules on the processors and the incurred intermodule reference costs. The problem can be formulated as a zero-one quadratic programming problem. The execution costs are expressed in the objective function simply as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} e_{ij}x_{ij}.$$

Communication costs incurred by modules not assigned to the same processor are of the form

$$\begin{aligned}
&\sum_{i=1}^{m-1}\sum_{j=1}^{n}\sum_{k=i+1}^{m} c_{ik}\,(1 - x_{ij})\,x_{kj} \\
&= \sum_{i=1}^{m-1}\sum_{j=1}^{n}\sum_{k=i+1}^{m} c_{ik}x_{kj} - \sum_{i=1}^{m-1}\sum_{j=1}^{n}\sum_{k=i+1}^{m} c_{ik}x_{ij}x_{kj} \\
&= \sum_{i=1}^{m-1}\sum_{k=i+1}^{m} c_{ik}\left(\sum_{j=1}^{n} x_{kj}\right) - \sum_{i=1}^{m-1}\sum_{j=1}^{n}\sum_{k=i+1}^{m} c_{ik}x_{ij}x_{kj}.
\end{aligned}$$

Since the summation in parentheses is one, and the summation of the c_{ik} is just a constant, the first group of communication terms can be ignored in the objective function. Thus the formulation of the problem is as follows:

minimize

$$\sum_{i=1}^{m}\sum_{j=1}^{n} e_{ij}x_{ij} - \sum_{i=1}^{m-1}\sum_{j=1}^{n}\sum_{k=i+1}^{m} c_{ik}x_{ij}x_{kj}$$

subject to the constraints

(c1) $x_{ij} = 0, 1$ for all i, j ($x_{ij} = 1$ if module i is assigned to processor j, and otherwise $= 0$)

(c2) $\sum_{j=1}^{n} x_{ij} = 1$ for all i (each module is to be assigned to exactly one processor)

where m is the number of program modules and n is the number of processors.

General zero-one polynomial programs can be converted to linear programs with non-linear secondary constraints [12, 32], however the problem at hand has been approached by using techniques which take advantage of the special structure of this problem.

The problem has been solved for $n = 2$ by Stone [29]. A model is developed that can be interpreted as a commodity flow network, and an assignment is made by applying a maximum flow algorithm [11]. Efforts to extend this method to the general n-processor case have not been completely successful.

For n-processor problems in which the intermodule reference pattern is constrained to be a tree, an optimal assignment can be obtained by using a shortest path algorithm [3]. The graph model developed for this restricted problem is extended in the next section to allow an arbitrary module intercommunication pattern.

Assignment algorithms such as the ones to be described here may be used to find a static assignment of modules to processors, but may also be applied repeatedly during the execution of a program to reassign modules dynamically as the program's working set changes. (The dynamic reassignment problem has been investigated by Bokhari [2].)

3. SHORTEST PATH ALGORITHM

A state-space representation of the problem [16, 22] is given by a graph in which the nodes represent program modules assigned to processors, and the arcs connecting the nodes represent communication between modules. The node $[ij]$ represents the assignment of module i to processor j. For a program of m modules to be assigned among n processors, the graph model will contain mn nodes. Arcs are defined and labeled as follows:

a. The arc from node $[ij]$ to node $[kj]$ is labeled with the value e_{kj}, the execution cost of module k on processor j. (Since both modules i and k reside at processor j, there is no communication cost to consider.)

b. The arc from node $[ij]$ to node $[pq]$ is labeled with the value $(e_{pq} + c_{ip})$, the sum of the execution cost of module p on processor q and the cost of communication between modules i and p.

c. The arc from node $[ij]$ to node $[ik]$ does not exist, since a module is assumed not to communicate with itself.

Directions are imposed arbitrarily on the arcs, in such a way that the graph is acyclic. This does *not* imply a precedence ordering among the program modules, but rather simply establishes a direction in which a search procedure is to advance through the graph model. The acyclic property of the graph is easily maintained as the graph model is built, by appropriately sorting the input data. The data is arranged so that arcs are always directed from an existing node to a node not yet in the graph. This insures that all arcs are oriented in a "forward" direction, and never point "back" in a manner that would generate loops in the graph model.

The set of nodes representing the possible assignments of a particular module to all processors is referred to as a *stage*, and the nodes within a stage are the possible *states* of that stage.

There is one stage (the first stage in the graph) which has no predecessor stage. A single artificial initial node s is added as a predecessor to this stage. The arcs from this initial node to the nodes $[ij]$ in the first stage are labeled with the cost of executing module i on processor j.

Several stages may have no successor stage, therefore a final node t is added to the graph model, to serve as a single termination point. Arcs from nodes in the terminal stages to this final node are all labeled with the value zero.

The model so constructed is called a *search graph.* The algorithm to be described evaluates the states in every stage of the search graph with the goal of selecting the best state for each stage (that is, making the optimal assignment of each module to a processor).

Figure 1 depicts a program of $m = 5$ modules which are to be distributed among $n = 3$ processors. The *program graph* consists of nodes representing modules and arcs representing the pattern of intermodule references. Arc labels indicate the intermodule reference cost incurred when the two modules run on different processors. These costs are shown in the communication cost matrix. The execution costs of the five modules on the three processors are given in tabular form in the execution cost matrix. Finally, the directed acyclic graph shows the directions arbitrarily imposed on the arcs in preparation for the construction of the *search graph.* The complete search graph is illustrated in Figure 2, where the large circles represent the three possible ways of assigning each of the five modules, and the arcs are labeled according to the rules specified at the beginning of this section.

A stage with multiple successor stages, such as that represented by module A in the figure, is called a "fork" stage. A "join" stage is one with multiple predecessor stages, such as module D in the figure.

The search algorithm is a node-labeling technique consisting of two phases, a labeling phase and an assignment phase:

Labeling Phase

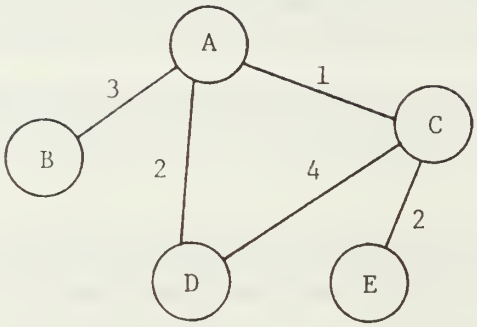
The procedure begins with the terminal node t and advances toward the initial node s, and labels all of the nodes according to the following rule.

Labeling Rule:

CASE 1: If the node has only one stage as successors, then label the node with the minimum (over all successor states) of the sum of the label on the successor node and the weight on the arc between the current node and the successor node.

CASE 2: If the node has more than one stage as successors, then compute a partial label (using the rule for Case 1) based on each successor stage, and add these values to obtain the label for the current node.

The label finally obtained for the initial node s is an upper bound on the cost of the assignment produced by the algorithm. (The relation between this upper bound and the actual cost will be discussed later in this section.)



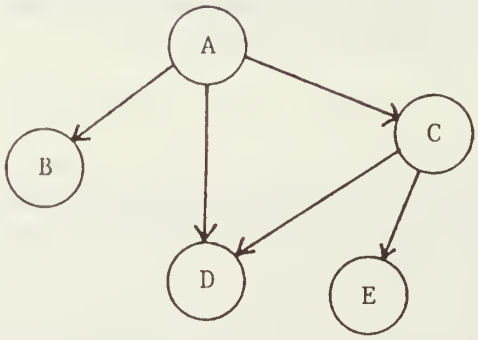
a) Program graph

	A	B	C	D	E
A	0	3	1	2	0
B	3	0	0	0	0
C	1	0	0	4	2
D	2	0	4	0	0
E	0	0	2	0	0

b) Communication costs

	P1	P2	P3
A	4	2	∞
B	3	1	2
C	4	∞	4
D	1	4	3
E	2	∞	3

c) Execution costs



d) Directed acyclic graph

FIGURE I. Modular program graph.

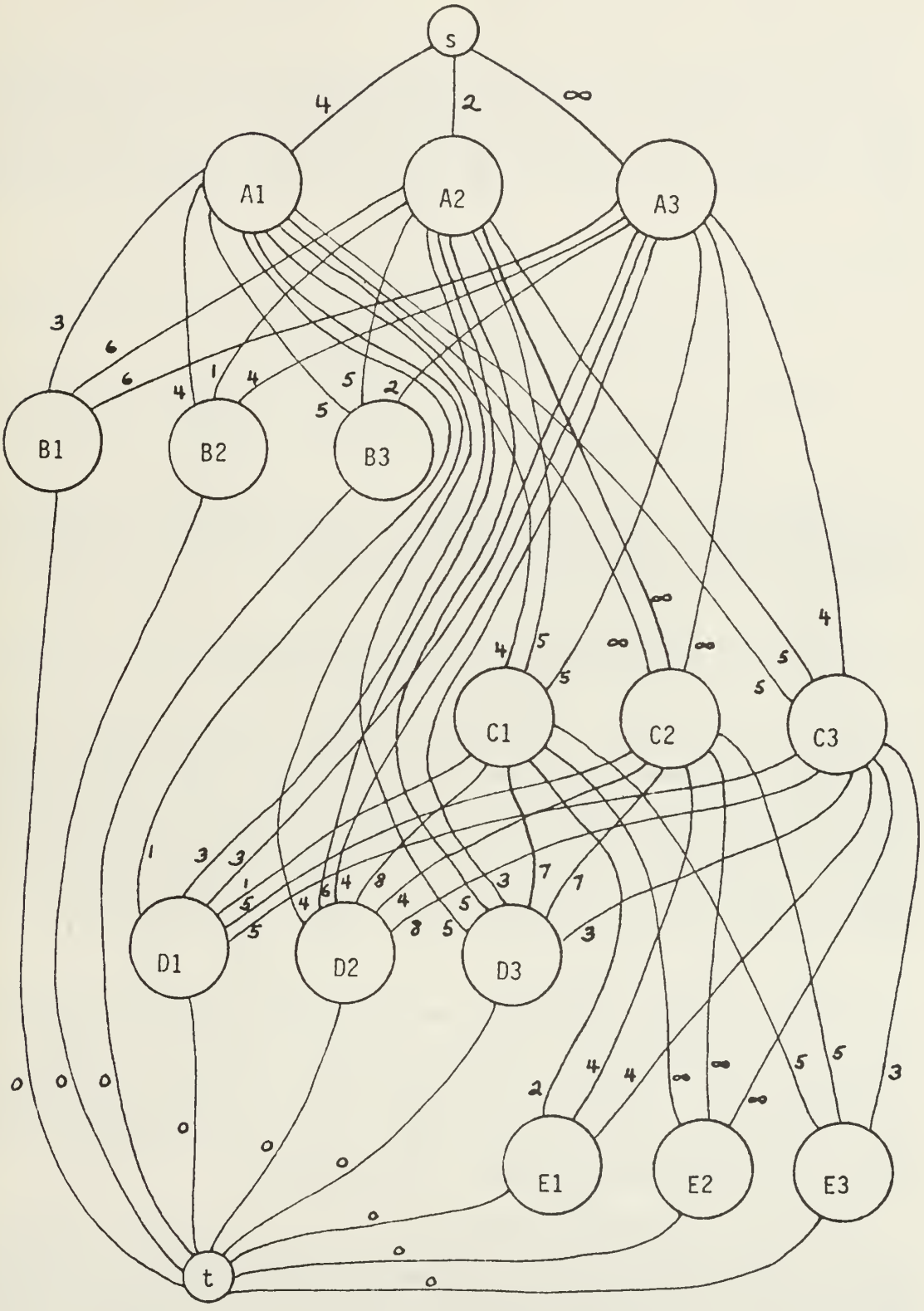


FIGURE 2. Search graph

Assignment Phase

Once the labeling process is complete, the actual assignment is obtained by starting with the initial node s, advancing through the search graph, and selecting at each stage the successor node, or nodes, used in the Labeling Phase to generate the label for the current node. The following rules apply to each node already "marked," that is, already selected to be in the assignment. (The initial node s is considered to be trivially a part of the assignment, so it is marked first.)

Assignment Rule:

CASE 1: If the node has only one successor stage, mark the successor node selected in generating the label for the current node.

CASE 2: If the node has more than one successor stage, mark the successor node in each successor stage that was selected to contribute to the label on the current node.

The assignment is complete when the terminal node t is marked. The subgraph of the original graph model, which consists of all the marked nodes and the arcs connecting them, constitutes the *assignment graph.* In Figure 3, the assignment graph is shown in bold lines and the node labels are written inside the circles denoting nodes.

The optimality of the assignment resulting from this search procedure is dependent upon the structure of the search graph, but is perhaps best analyzed as follows.

Node labeling is begun at the terminal node and advances backward until the initial node is labeled; thus the procedure is known as *backward analysis* or *backward recursion.* The underlying dynamic programming concept is known as the dynamic programming principle of optimality [1, 9, 15]: that the optimal policy for the remaining stages (from the current node to the terminal node) is independent of the policy adopted in previous stages (from the initial node to the current node). At each stage, the label for each state denotes the minimum cost of any sequence of decisions that can be made from the current state to the terminal node. The algorithm therefore produces an assignment at each stage by evaluating all possible states in that stage and selecting the most attractive.

When a node n_i has successors n_j, for $j = 1, \ldots, n$, which all belong to one stage, then this represents a communication between just two program modules. It suffices to label node n_i with the minimum of the sum of the weight on the arc to n_j and the label on node n_j, since this constitutes the optimal decision at node n_i. The weights on the arcs are such that the intercommunication costs are included between modules on different processors, and not when two communicating modules reside on the same processor. Thus the label on node n_i is the optimal cost of communicating with and executing the module represented by n_j plus the cost of completing all remaining modules. This is Case 1 of the Labeling Rule.

When node n_i has successors n_j belonging to more than one stage (i.e., when node n_i represents a "fork" stage), then the module represented by n_i is communicating with more than one other higher-numbered module, and *all* of these costs must be considered. Case 2 of the Labeling Rule sums the minimized costs for all successor stages and labels node n_i with the result.

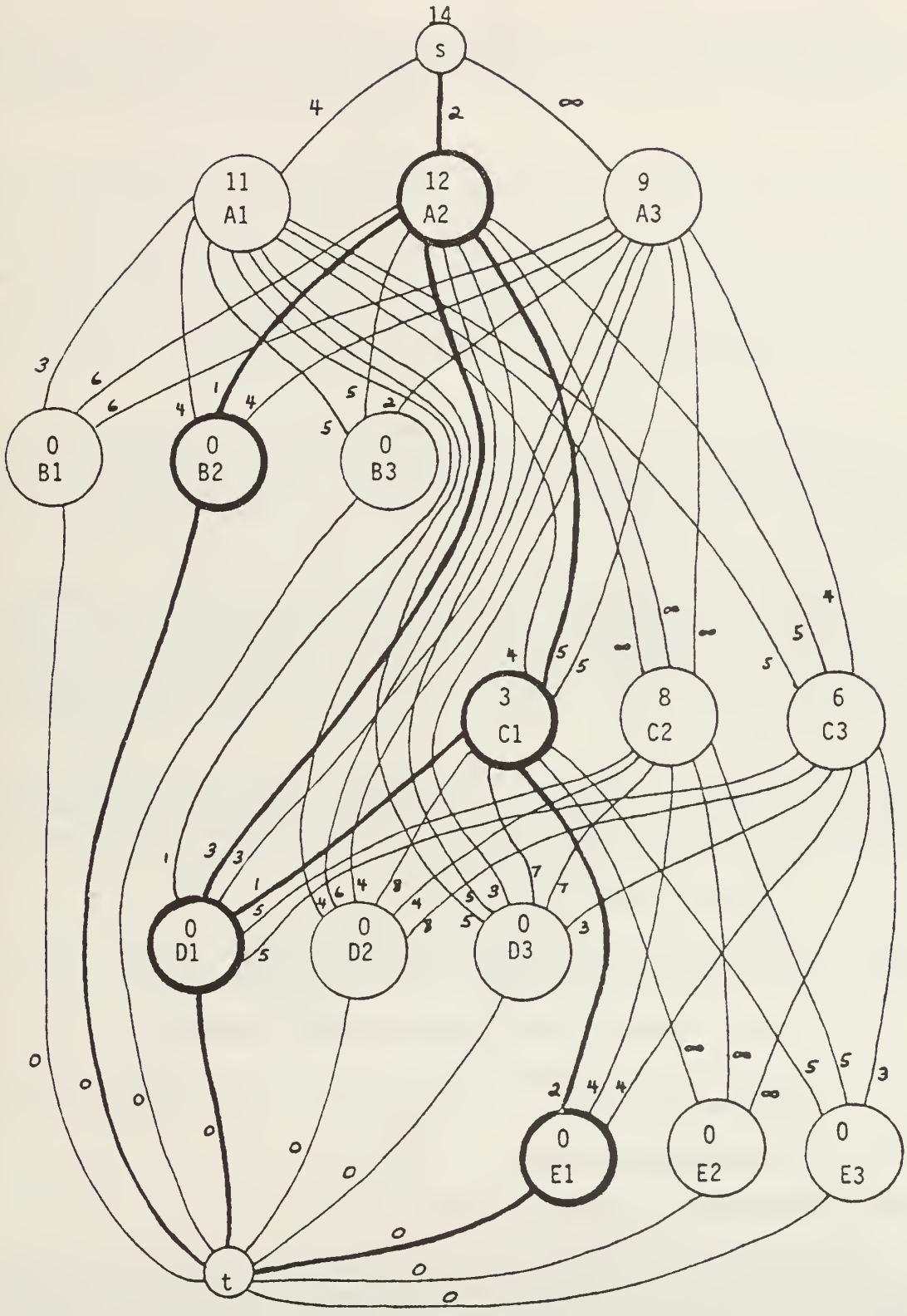


FIGURE 3. Assignment graph

If the module intercommunication pattern is a tree rooted at the initial node s, then an optimal assignment always results from the algorithm. The label on node s is equal to the cost of the optimal assignment, and the assignment itself is obtained during the Assignment Phase of the algorithm.

Because the search graph is not generally a tree, and more than one stage, say B and C, may share the same successor stage, say D, the cost from stage D to the terminal node will be included in the labels for both stages B and C. As the procedure advances upward from B and C to a common predecessor stage, say A, the labels assigned to nodes in stage A will be computed in such a way that the costs from successor stage D downward are counted twice. Thus, in this case, the label on a node actually represents an upper bound on the cost from that node to the end of the search graph. (It should be noted that the occurrence of an inflated label at any node of the search graph does not prevent the algorithm from yielding a solution that is guaranteed to be optimal.)

A couple of practical considerations are addressed at this point. First, in a general search graph, "ambiguous" assignments may arise; a method will be given here for resolving such ambiguities. This is followed by a discussion of the second problem: that encountered in cyclic search graphs. A technique will be described which guarantees the construction of acyclic models.

In the problem illustrated in Figure 3, module D is assigned to processor 1 because the node [D1] is the optimal successor to both node $[A2]$ and node $[C1]$ and thus no conflict occurs. In contrast, an ambiguous situation is illustrated by the assignment graph in Figure 4, in which the optimal successor to node $[B1]$ is $[D1]$, but the optimal successor to node $[C1]$ is $[D2]$. Such an ambiguity must be resolved in order to obtain a valid assignment, satisfying constraint (c2).

Let X and Y be predecessor stages of stage Z. Suppose that $[Xi]$ and $[Yj]$ are optimal assignments, and that $[Zk]$ is the optimal successor to $[Xi]$, and $[Zl]$ is the optimal successor, to $[Yj]$. Let $\lambda(Al)$ denote the label on the node $[Al]$. Then ambiguities are easily resolved by selecting the state at stage Z corresponding to the

$$\min_{Zk,Zl} \{e_{Zk} + c_{XZ} + c_{YZ} + \lambda(Zk),\ e_{Zl} + c_{XZ} + c_{YZ} + \lambda(Zl)\}.$$

In general, a stage Z may have many predecessor stages, each suggesting a different assignment at stage Z. In that case, each possibility must be evaluated by taking the sum of:

1. the execution cost of the module associated with the current state,
2. the sum of the communication costs between the current state and all its assigned predecessor states, and
3. the label on the current state.

The least costly alternative is then selected.

It happens that, for the graph in Figure 4, an optimal assignment results from applying the above rules for resolving ambiguities. However, in general, when ambiguities arise, the optimality of the assignment is not guaranteed. If D is any "join" stage (such as in Figure 4)

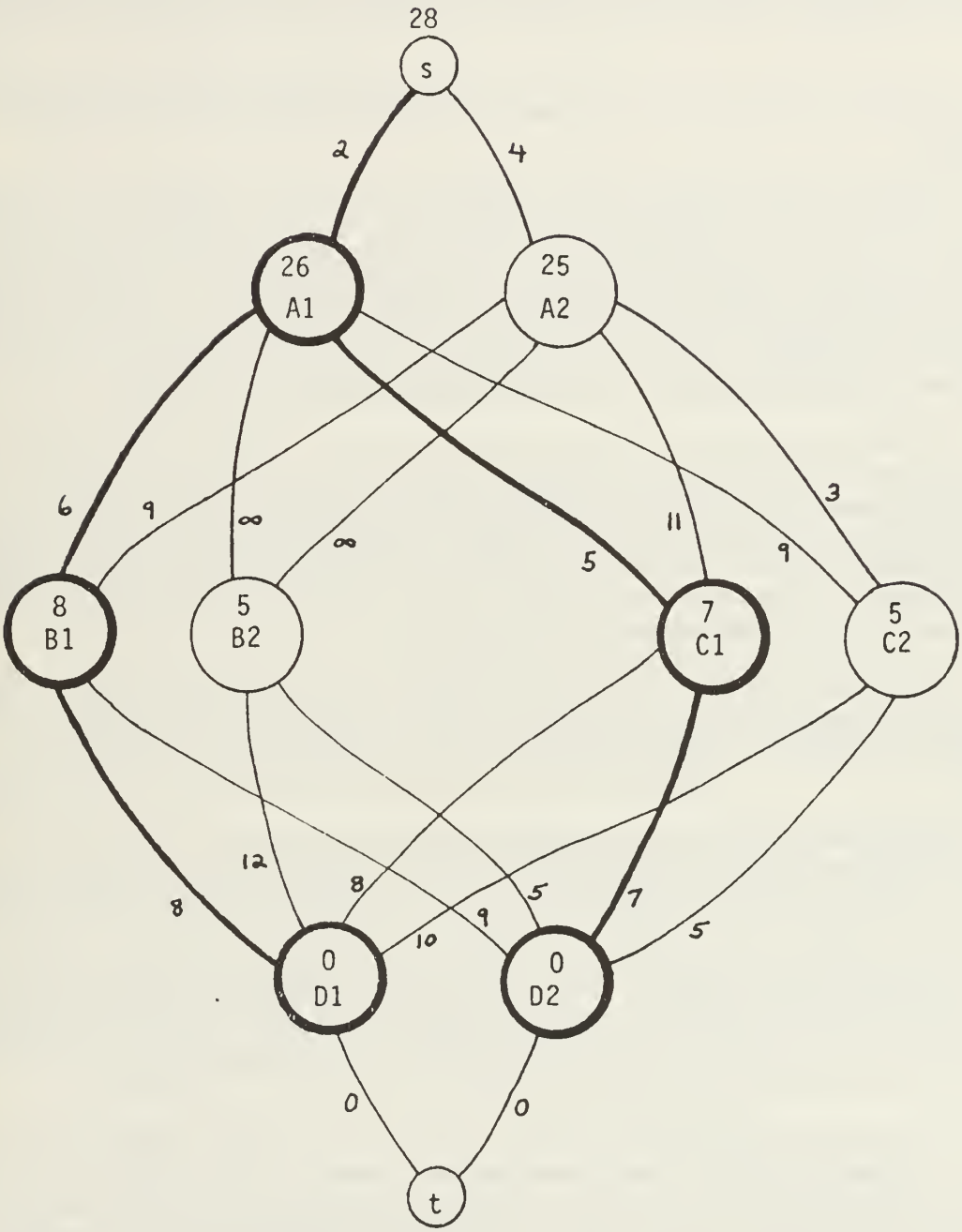


FIGURE 4. Ambiguous assignment

and A is any preceding "fork" stage, then the label on a node in stage A may contain incompatible constituents. In the cited example, the left constituent is based on state 1 in stage D, while the right constituent is based on state 2 in stage D. Clearly, constraint $(c2)$ does not allow both states in the assignment, therefore the labels on nodes in stage A are not necessarily realistic.

Note that the presence of a "fork" and "join" substructure within the search graph does not necessarily give rise to ambiguities and therefore need not compromise the optimality of the assignment.

The effect of the structural properties of the search graph on the optimality of the solution produced by the shortest path algorithm described in this section may be summarized as follows. The graph may represent intermodule reference patterns that are trees or not trees. In those that are trees, ambiguities never arise and optimality is guaranteed. In those that are not trees, ambiguous assignments may or may not arise. If ambiguities do not arise, optimality is guaranteed. In ambiguous situations, an optimal assignment may or may not result. (The algorithm does identify those ambiguous situations for which optimality is obtained.)

The solution approach used in this section is basically a shortest path method, and it should be noted at this point that the problem of resolving ambiguities is the primary motivation for the use of a backward shortest path method rather than a forward method such as Dijkstra's labeling algorithm [7, 8, 10]. Forward and backward methods differ in the auxiliary information they produce. The backward procedure described in this section generates explicit information about optimal paths from particular nodes to the terminal node and this is precisely the information used in the resolution of ambiguities.

The search graph model which has been described is an acyclic directed graph. If cycles were allowed in the search graph, two difficulties could arise. One of the problems could be solved by a modification to the algorithm, but the other might cause the procedure not to terminate.

The first problem is that a node may be assigned a label at one point, and later, since it is its own predecessor, receive a second, larger label. The labels on the nodes in any simple path from the terminal node to the initial node in an acyclic search graph form a partially ordered chain, since if node $[AI]$ is an immediate predecessor of node $[BJ]$, then $\lambda(AI) > \lambda(BJ)$ by an amount equal to e_{BJ} if $I = J$, and by an amount equal to $(e_{BJ} + c_{AB})$ if $I \neq J$. This property is lost if cycles are allowed. A node in a cycle may be labeled twice, and the final label will in general be larger than the labels on the node's predecessors in the cycle. The most recent label is the one that should be permanently assigned to that node; however care must be taken not to label a chain of predecessors of a node more than once in the algorithm.

The more serious problem arises when a module having multiple predecessor stages is given an ambiguous assignment, during the assignment phase, and is involved in a cycle. In this case, in the evaluation of a particular alternative state in the ambiguous stage, the label of the state may be dependent upon selection of a *different* predecessor state at that same stage. Thus the evaluation process may require indefinite backtracking through the cycle, in an attempt to resolve an unresolvable ambiguity.

Cyclic search graph models are avoided simply by arranging the input to the algorithm so that the modules are introduced in topological order. Information about the execution and

communication costs of module i is introduced before the information about module j if and only if $i > j$. Thus the graph model is constructed, beginning with the terminal node, and directions are imposed on the arcs in such a way that an arc is never directed from a node in stage j to a node in stage i unless $i > j$.

4. BRANCH-AND-BOUND ALGORITHM

The algorithm described in the preceding section generally yields low-cost assignments of modules to processors, but has two characteristics which become serious disadvantages when the search graph is large, that is, when modules of a very large distributed program are to be assigned among processors in a large network. First, an evaluation (in the form of a computed label) is made of every node in the graph, and, second, the entire search graph must be maintained in memory to allow a final assignment to be made. It is therefore reasonable to attempt to modify the technique so that it is less costly in its consumption of time and memory space, yet still produces a reasonably low cost assignment of modules to processors. This compromise represents an effort to minimize the combined expense of the assignment and the process used to obtain the assignment.

The graph model to be used as the search graph is the same one described in the previous section. A forward search procedure is given that begins with the initial node and searches selected portions of the graph to obtain a solution. The following three considerations contribute to the efficiency of the technique:

a. An evaluation function is applied to the nodes in order to estimate the "promise" of a node. This criterion is used to direct the search to the nodes thought to be best.

b. Not all of the successors of a node are generated. Since it is unlikely that an optimal assignment would place a module on a processor poorly suited for it, only those nodes representing assignments with lower execution costs are considered.

c. In order to reduce the total storage requirements, final assignment decisions are made within certain regions of the graph, thus allowing upper parts of the search graph to be discarded and making space available for use by deeper parts of the graph.

The heuristic procedure used here is a nonbacktracking branch-and-bound procedure that uses a local neighborhood search at each stage [21, 25]. An important characteristic of this procedure is that a feasible (and always improving) solution is available at all times, in case early termination of the assignment algorithm is necessitated.

The following definitions will be used in describing the algorithm.

π denotes any valid assignment graph (representing any feasible assignment).

S is the solution space, defined as {all π}.

π_i denotes a partial assignment graph in which modules 1 through i have been given permanent assignments and the remaining modules have tentative assignments.

π_{ik} is the partial assignment π_i in which module i is assigned to processor k.

π_m represents a complete assignment graph, where m is the number of modules.

$L(\pi_i)$ denotes the cost of a partial assignment π_i in which communication costs are considered for module 1 through module i but not for modules $i+1$ through m.

$f(\pi_i)$ denotes the total cost of execution and communication for a partial assignment π_i.

$f(\pi_m)$ denotes the total cost of a complete assignment of all m modules, that is, the total cost of executing the distributed program for the assignment π_m.

$N(\pi_{ij})$ represents a neighborhood of π_{ij} defined as

$$\{\pi_{ik} \mid e_{ik} \leqslant \text{median of } e_{ij} \text{ for } j = 1, \ldots, n\}.$$

The neighborhood of a partial assignment is therefore simply that set of partial assignments π_i obtained by varying the decision at stage i among that half of the states for which the execution cost of the module i is least.

Note that $L(\pi_i)$ is a lower bound on the cost function f since

$$L(\pi_i) \leqslant f(\pi_i) \text{ for } 1 \leqslant i \leqslant m-1$$

and

$$L(\pi_m) = f(\pi_m).$$

If j is a successor stage of i, then

$$L(\pi_j) \geqslant L(\pi_i)$$

therefore the lower bound increases as the search proceeds through the graph.

The algorithm utilizes the lower bound $L(\pi_i)$ for $i = 1, \ldots, m$. Heuristic information may be used to establish this bound since $f(\pi_i)$ is generally not known until the search is complete. It is desirable that a lower bound be as close as possible to the actual cost function $f(\pi_i)$, but it is also necessary that L be simple to compute or estimate.

The bounding strategy selected here is the use of a modified objective function in which the execution costs (the linear terms) are considered but the communication costs (the quadratic terms) are ignored until they are easily available. Thus, initially, the lower bound value used is

$$L(\pi_1) = \sum_{i=1}^{m} (\min_j \{e_{ij}\})$$

The original problem constraints are not relaxed, so the initial lower bound is just the cost of executing each module on the processor for which its execution cost is least. As communication costs are encountered at each stage in the search graph, they are included in the lower bound L; that is, terms are added to the objective function to yield a more realistic assessment of the cost. Finally, the original complete objective function is generated and $L(\pi_m) = f(\pi_m)$.

A formal description of the algorithm is given in Appendix A. The notation appears at first to be slightly cumbersome, but is justified since it is a simplification of a broader scheme [18, 19] used to characterize a general class of branch-and-bound algorithms, and therefore

allows the technique to be defined in a standard framework. It should be clear that the algorithm can be stated quite simply in terms of the search graph defined in Section 3. The search begins at the initial node s and at each stage selects the state indicated by taking the arc with the smallest label. Ambiguities are resolved, as in Section 3, on the basis of local information (i.e., information at the current stage and immediate predecessor stages).

Several considerations pertinent to a practical implementation of the algorithm are mentioned here:

A table of module execution times is created before the search begins. This information is used to establish the initial solution which is based solely on execution times. Module intercommunication costs are made available to the algorithm as the search progresses.

A list of "active" modules may be maintained that contains information about modules that have been generated by the Branching Rule but not yet abandoned by the Deletion Rule.

Implementation of the policy Q for selecting the partial assignment π_{jl} such that

$$L(\pi_{jl}) \leqslant L(\pi_{jk}) \text{ for } all\ \pi_{jk} \in N(\pi_{j1'})$$

is accomplished simply. Note that the values of all the $L(\pi_j)$ differ only in the execution costs e_{js}, where $s = 1', \ldots, k'$ and $k' = n/2'$, and the communication cost c_{ij} where i is the predecessor stage of state j. Thus, the least lower bound is selected by considering the states $j1'$ through jk' and choosing the one corresponding to

$$s = \min_{1', \cdots, k'} \{e_{js} + c_{ij}\}.$$

Ties may be broken arbitrarily.

Figure 5 illustrates the result of applying this technique to the search graph shown in Figure 2. It happens in this case that the branch-and-bound algorithm does produce an assignment that is optimal.

5. COMPUTATIONAL RESULTS AND SUMMARY

Implementations of the two algorithms described in Sections 3 and 4 (plus an algorithm which performs an exhaustive enumeration of all possible assignments) were developed for performance analysis. Random test data for experimentation were systematically generated for hypothetical networks of 5, 10, 15, and 20 processors and 5, 10, 15, and 20 program modules. Nine different networks were generated for each problem size (m, n). Of interest are the computation time requirements for each problem and the cost (i.e., objective function value) of the assignment produced. Average computation times for the algorithms are reported in Table 1.

The shortest path labeling algorithm requires roughly n^2m additions and $(n-1)nm$ comparisons (if no ambiguities occur) and therefore should run in $0(n^2m)$ time. However, in most of the test cases, ambiguities did arise and computation time on these sample networks was $0(n^2m^2)$.

The nonbacktracking branch-and-bound search requires nm additions and $(n-1)m$ comparisons, and, as expected, the computation time exhibited on the test cases was $0(nm)$.

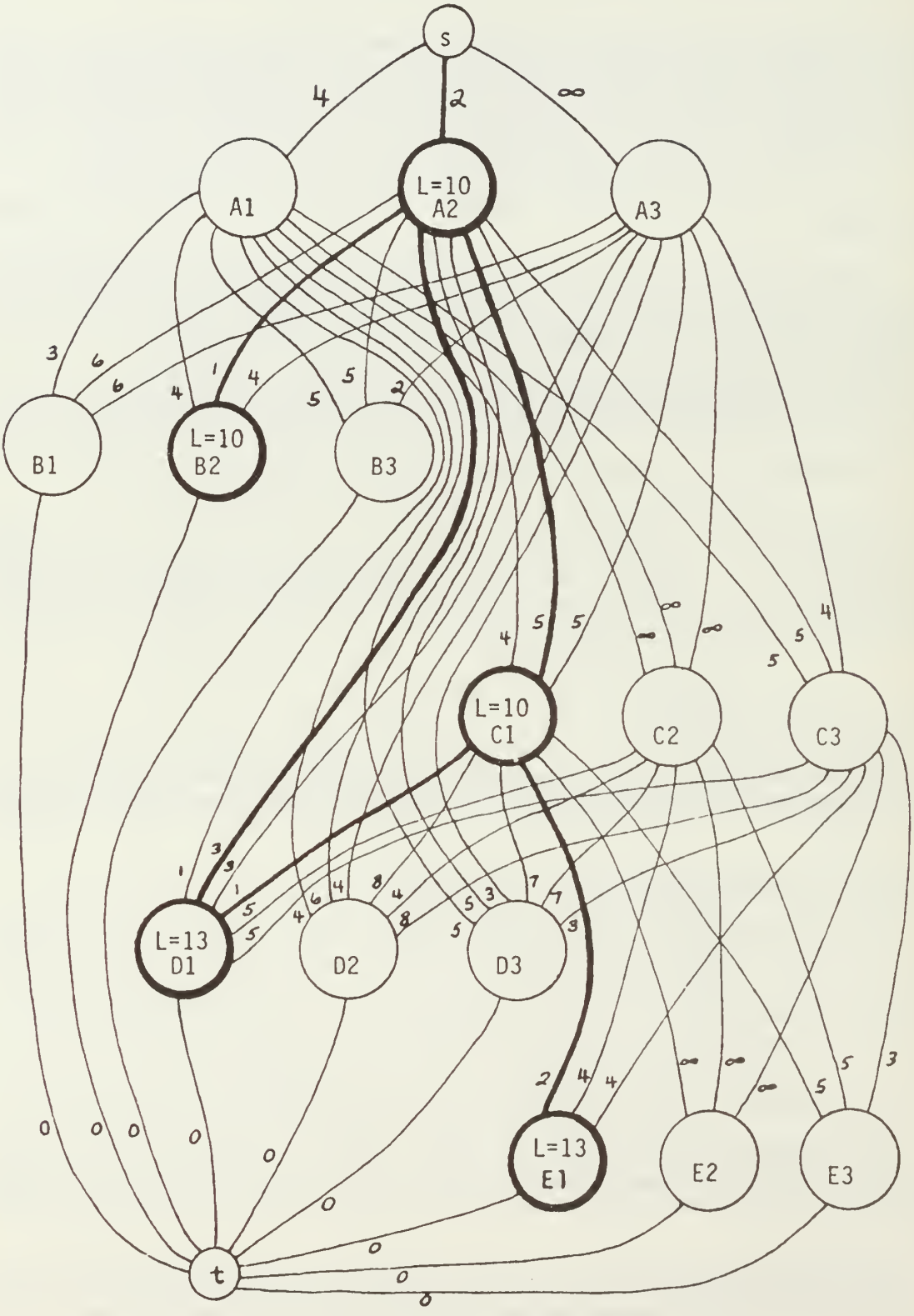


FIGURE 5. Branch-and-bound assignment graph.

TABLE 1 — *Computation Times*

	$N = 5$	$N = 10$	$N = 15$	$N = 20$
$M = 5$	.0156 .002	.0400 .008	.0727 .0102	.1220 .0216
$M = 10$	.058 .0083	.189 .0152	.373 .0304	.654 .044
$M = 15$	.142 .0124	.443 .030	.938 .048	1.61 .074
$M = 20$	.266 .022	.835 .042	1.68 .068	2.95 .103

For each problem size, the computation time for the shortest path algorithm is shown above the time for the branch-and-bound algorithm.

Computation times are given in seconds.

The quality of the solutions produced by the two algorithms varies depending on the nature of the network. In sample networks having execution costs and communication costs of the same order of magnitude, the shortest path algorithm outperforms the branch-and-bound method slightly. The difference in performance is less pronounced in networks in which execution costs dominate communication costs. In these cases, the cost of the assignment produced by these algorithms never exceeds twice the cost of an optimal assignment. However, when communication costs greatly exceed execution costs, both algorithms perform poorly (in some cases yielding a cost as high as 5 times optimal cost) and the branch-and-bound algorithm, of course, is the worse of the two. This performance is not surprising in a network dominated by communication costs. Recall that in the objective function of the problem formulation (section 2), communication costs give rise to the nonlinear terms, and it is precisely the nonlinearity of the objection function that makes this particular scheduling problem a difficult one to solve exactly.

There have been no published reports of computational experiments with other approaches to this particular scheduling problem, so there is no information available on which to base estimates of relative efficiency.

The scheduling problem considered here has an efficient solution for $n = 2$ [29]. It has been shown [30] that the problem is *NP*-hard for $n \geqslant 4$. This property has not been established for the 3-processor case.

There are n^m possible assignments to be considered in this scheduling problem and, indeed, actual computational experiments using an enumerative algorithm require time that is $0(n^m)$. The shortest path and branch-and-bound algorithms are of low polynomial complexity but generally produce suboptimal solutions.

The problem of assigning modules of a computer program among processors in a distributed computer network has been investigated and formalized here as a zero-one quadratic programming problem with linear constraints. This particular scheduling problem has received relatively little attention, and several interesting topics remain for further study.

a. Some effort should be directed to studying the way in which the execution costs e_{ij} (which represent affinity values for module i to processor j) are determined. A reasonable approximation to these costs may be obtained by assuming that e_{ij} is some fixed constant times e_{ik}, where the fixed constant is some predetermined value which relates the overall speed and processing power of processors j and k. This, of course, ignores the nature of module i. Some experiments might be performed to show how additional information may be gathered (for example, from previous runs) to refine initial estimates of relative processor performance.

b. A similar analysis ought to be made for the problem of establishing intermodule reference costs. Initially such costs might be assigned based solely on the number of intermodule references, assuming that each reference incurs the same cost. However, since a control program must intervene to perform intermodule transfers, information could be gathered to improve the original estimates.

c. Special problems arise in the determination of execution costs if modules are allowed to call themselves recursively, and also in the determination of communication costs if a nondeterministic referencing environment is permitted (i.e., if address calculations are allowed during execution). A consideration of both of these issues introduces an element of program data dependency which complicates the model.

d. A module should not, in practice, be assigned to a processor that is overloaded, even though that processor may be the one best suited for the module [28]. Processor load balancing might be included in the branch-and-bound procedure through a modification of the Branching Rule. A neighborhood could be redefined so that it is based not just on execution times, but rather on some combination of execution costs and processor availability. A workable measure of current processor load must be developed which can be incorporated into the existing model.

The search for a solution to this problem has included a consideration of various techniques such as multi-terminal cut trees [13], spanning trees [20], and cluster analysis [34], in addition to the approaches described in the preceding sections. Scheduling problems generally rely on a variety of techniques from the areas of linear and nonlinear programming, network analysis, and graph theory. It appears that until fundamental theoretical advances are made, practical developments will consist of heuristic methods and combinations of algorithms from diverse fields.

APPENDIX A

The nonbacktracking branch-and-bound algorithm can be characterized by a seven-tuple

$$(I, B_N, S_Q, L, P, D, R)$$

for which parameters are defined as follows:

1. *Initial Solution Rule.* An initial solution is obtained by assigning each module to the processor for which its execution cost is least.

2. *Branching Rule.* The branching process extends a partial assignment π_i into partial assignments π_{jk} where j is a successor stage to i and $k = 1', \ldots, n/2'$, and nodes $j\,1'$ through

$j\ n/2'$ represent that half of the states in stage j having the lowest exectution costs. Thus, an application of this rule generates the neighborhood $N(\pi_{j1'})$, where $j1'$ is the initial state at stage j. The branching rule must be applied to all successor stages of the branching node in stage i.

3. *Selection Rule.* The selection of the next branching node is based upon a search within the neighborhood $N(\pi_{ij})$ of the state representing the initial assignment of module i to processor j. The policy Q for making the selection is to choose that π_{il} such that

$$L(\pi_{il}) \leqslant L(\pi_{ik}) \text{ for all } \pi_{ik} \in N(\pi_{ij}).$$

4. *Lower Bound Function.* A lower bound on the cost of partial assignment π_i is $L(\pi_i)$, as defined in the list of definitions in Section 4.

5. *Permanent Assignment Criterion.* If the current stage i has predecessor stages which do not have a permanent assignment, then the assignment of the current stage remains tentative even though the selection rule has been applied and an assignment decision made. At a "join" point, a selection process must be applied to the stage for every predecessor stage, and ambiguities resolved if necessary. Only then may the assignment at stage i be considered permanent and the assignment π_i established.

6. *Deletion Rule.* A stage i may be deleted from the search graph when all successor stages have permanent assignments, that is, when π_k has been established for all successor stages k of stage i.

7. *Resource Limitations Vector.* The components of this vector may be used to control the extent of the search, by giving an indication of the availability of the resources required during the search. An upper bound on algorithm execution time may be checked and the algorithm halted if time is exceeded. An upper bound on available storage may be used to vary the breadth of the search at each stage. The size of the neighborhood $N(\pi_{ij})$ generated by the Branching Rule may be increased or decreased appropriately.

ACKNOWLEDGMENTS

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LEAST-ABSOLUTE-DEVIATIONS POSITION FINDING

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ABSTRACT

Position finding has historically been carried out by calculating the coordinates of the mean position via a least-squares procedure based on the distance of the position from several direction lines. It has been suggested that the least-squares procedure assigns too much weight to outliers among the set of direction lines, outliers which may actually be associated with objects other than the one being located. In this paper, a method of using least-absolute deviations, which yields a more outlier-resistant median estimate of the position instead of the least-squares mean estimate, is presented.

1. INTRODUCTION

Position finding is the process of locating an object on the earth's surface by amalgamating the observation (or "direction") lines to the supposed position made by various observers scattered distances from the true position. If all observers could construct their direction lines without error, all lines would theoretically intersect in the true position. In reality, of course, the direction lines have a certain amount of error associated with them, and this circumstance necessitates a statistical treatment of the subject.

The statistical foundations of position finding were developed in 1951 by H.E. Daniels [1]. Daniels presented the following assumptions as valid in most cases of practical interest:

(i) the earth is flat near the true position;

(ii) the direction lines are straight lines;

(iii) observation errors displace the direction line parallel to itself.

Daniels represented the equations of the direction lines in terms of an angle of inclination ϕ and the perpendicular distance p of the line from the origin. In line with current practice (see Figure 1) this makes the equation of the direction line

$$-x \cos \phi + y \sin \phi = p.$$

So, in general, we have a sequence of N direction lines, one equation $-x \cos \phi_j + y \sin \phi_j = p_j$ for each value of $j = 1, 2, \ldots, N$. If we denote the true location of the object in question by (ξ, η), then the perpendicular distance from the point (ξ, η) to the jth direction line is $|\xi \cos \phi_j - \eta \sin \phi_j + p_j|$. Assuming that the parallel displacement error of the jth direction line is a random variable with mean 0 and variance σ_j^2, Daniels defined the "least-squares fix"

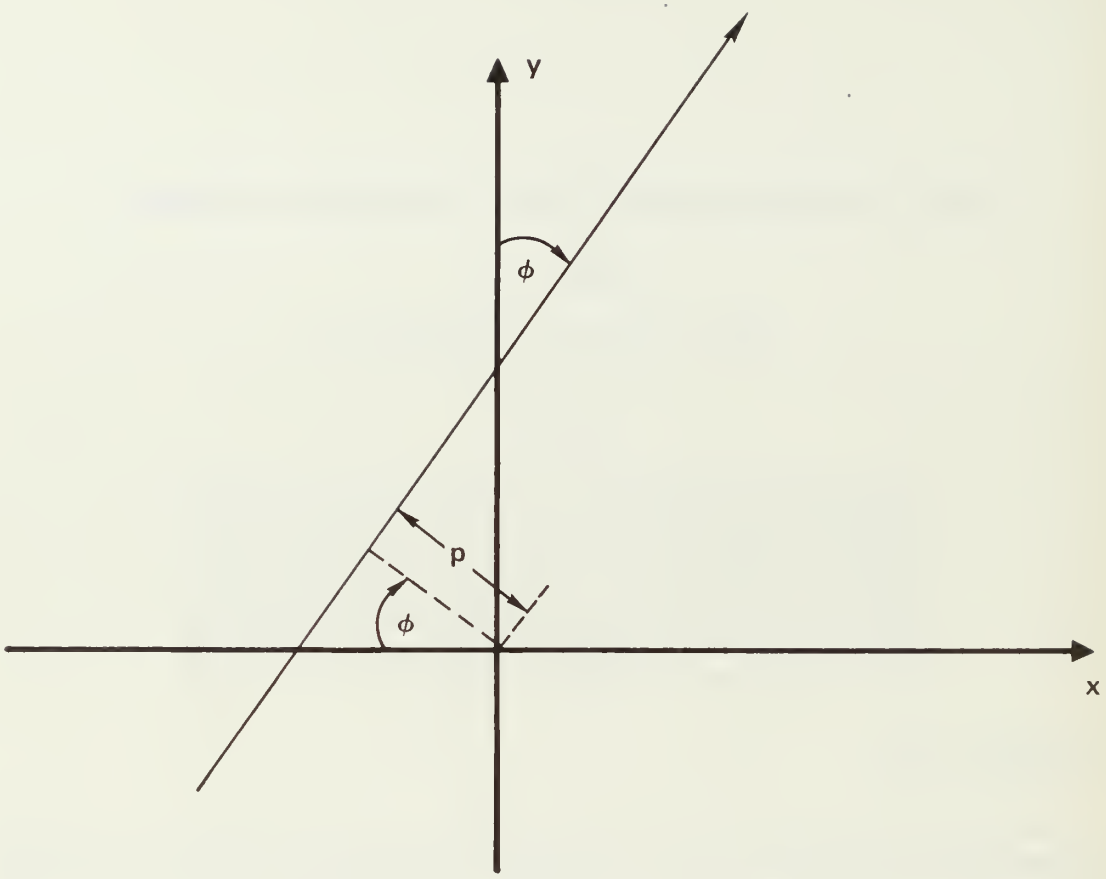

FIGURE 1. The direction line $-x \cos \phi + y \sin \phi = p$

to be the point (X^μ, Y^μ) that minimizes the sum of standardized squared perpendicular distances

$$E^\mu(x, y) = \sum_{j=1}^{N} \sigma_j^{-2}(x \cos \phi_j - y \sin \phi_j + P_j)^2.$$

We divide the jth squared distance by σ_j^2 because we want the least erroneous direction lines to count most heavily in the sum. The least-squares fix has the desirable property that, if all the σ_j's are equal, the vector sum of distances from (X^μ, Y^μ) to each of the direction lines is 0. Furthermore, "normal equations" and explicit formulas for X^μ and Y^μ can be found by the usual partial differentiation techniques. See Daniels [1] for further details.

Recently, it has been suggested that the least-squares procedure assigns too much weight to outliers among the set of direction lines, outliers which may actually be associated with objects other than the one being located. Such misassociations are common when two or more objects are relatively close together. Suppose, for example, that a set of 100 direction lines ostensibly fixed on object A actually contains 20 which are fixed on object B. The least-squares approach weights all 100 threesomes (ϕ_j, p_j, σ_j) equally and does not attempt to isolate the 20 outliers. Because the least-squares fix is essentially a mean of all 100 direction lines, we should not be surprised if the fix were located 20% of the way from object A to object B along the line joining them. We would then be making a possibly serious error in asserting that object A was

located at the least-squares fix. Our objective in this paper is to present a method of reducing the effect of outliers by using least-absolute deviations rather than least-squared deviations. This approach yields a median estimate of the true position which is more outlier-resistant than the traditional least-squares mean estimate.

To find the "least-absolute-deviations fix" (X^*, Y^*), we have to minimize the sum of standardized perpendicular distances

$$E^*(x, y) = \sum_{j=1}^{N} \sigma_j^{-1} |x \cos \phi_j - y \sin \phi_j + p_j|.$$

The fact that the absolute value function is not differentiable in a neighborhood of its minimum makes the usual approach impossible. Partial differentiation techniques alone cannot be used to minimize $E^*(x, y)$, and therefore, there are no normal equations and no explicit formulas for X^* and Y^*. Where do we begin?

We turn to the regression literature for ideas on how to proceed. There is a large number of papers dealing with "least-absolute-deviations regression." Examples are those of Singleton [13], Karst [6], Sharpe [11], Forsythe [3], and Schlossmacher [10]. Absolute-deviation minimization problems have been expressed and solved in terms of linear or convex programming by Wagner [14], Davies [2], Rao and Srinivasan [9], and Narula and Wellington [7], among others. Various competing methods are compared in Harvey [4], Hill and Holland [5], and Narula and Wellington [8]. Unbiased estimation is discussed in Sielken and Hartley [12].

In two dimensions, the least-absolute-deviations regression problem is to minimize

$$E^*(m, b) = \sum_{j=1}^{N} |y_j - mx_j - b| \tag{1}$$

while the position-finding problem is to minimize

$$\begin{aligned} E^*(x, y) &= \sum_{j=1}^{N} |x\sigma_j^{-1} \cos \phi_j - y\sigma_j^{-1} \sin \phi_j + \sigma_j^{-1} p_j| \\ &= \sum_{j=1}^{N} \sigma_j^{-1} \sin \phi_j |x \operatorname{ctn} \phi_j - y + p_j \csc \phi_j| \end{aligned} \tag{2}$$

which can be viewed as a "weighted" least-absolute-deviations regression problem.

If the observation errors are distributed according to the normal distribution (and there are no outliers other than those anticipated by this assumption), then Daniels [1] noted that the least-squares fix is the maximum likelihood estimator of the true position. While the major focus of this paper is the problem of unanticipated outliers, it is perhaps of some interest to mention a maximum likelihood property of the least-absolute-deviations fix. Suppose we set

$$\epsilon_j = x \cos \phi_j - y \sin \phi_j + p_j$$

to be the observation error in the jth position line, and we assume that ϵ_j has the Laplace (or "double exponential") distribution with mean 0 and variance σ_j^2. Then ϵ_j has the probability density function

$$f_j(\epsilon) = \frac{1}{\sigma_j \sqrt{2}} \exp\{-|\epsilon|\sqrt{2}/\sigma_j\}.$$

If there are no outliers other than those consistent with the Laplace distribution, the maximum likelihood estimator of the true position is obtained by maximizing the log-likelihood function

$$L(\epsilon_1, \epsilon_2, \ldots, \epsilon_N) = -\sum_{j=1}^{n} \log(\sigma_j\sqrt{2}) - \sqrt{2} \sum_{j=1}^{N} \sigma_j^{-1}|\epsilon_j|.$$

To maximize this is to minimize the second term (the only term containing x and y). But this is equivalent to minimizing Equation (2). Therefore, the least-absolute-deviations fix is the maximum likelihood estimate of the true position if the observation errors have the Laplace distribution.

2. THE MINIMIZATION PROCESS

Referring to the expression $E^*(x, y)$ in Equation (2), we define

$$E_j^*(x, y) = x \operatorname{ctn} \phi_j - y + p_j \csc \phi_j. \tag{3}$$

Without loss of generality, we may choose our origin of coordinates in such as way that, for each $j = 1, 2, \ldots, N$,

$$0 \leqslant \phi_j \leqslant 180°$$

$$p_j \geqslant 0. \tag{4}$$

If we fix x and consider $E_j^*(x, y)$ to be a function of y alone, then $E_j^*(x, y)$ represents a convex broken line in the (y, E^*) plane with a single vertex at the point

$$y_j(x) = p_j \csc \phi_j + x \operatorname{ctn} \phi_j, \quad E^* = 0. \tag{5}$$

Because $\sin \phi_j \geqslant 0$ by Equation (4), the absolute error

$$E^*(x, y) = \sum_{j=1}^{N} \sigma_j^{-1}(\sin \phi_j)|E_j^*(x, y)| \tag{6}$$

is a sum of broken lines and is therefore a piecewise-linear convex function of y, having N vertices, one vertex lying above each of the horizontal axis points of Equation (5). See Figure 2 for an illustration.

Because $E^*(x, y)$ is a piecewise-linear convex function in y for fixed x, it has a minimum which occurs at one of the points $y_j(x)$ of Equation (5). Its partial derivative with respect to y, namely

$$\frac{\partial E^*}{\partial y} = -\sum_{j=1}^{N} \sigma_j^{-1}(\sin \phi_j) \operatorname{sgn} E_j^*(x, y) \tag{7}$$

exists almost everywhere (y), in fact, everywhere except at those values of y in Equation (5) for which $\operatorname{sgn} E_j^*(x, y) = 0$. Here

$$\begin{aligned} \operatorname{sgn} \theta &= \quad 1 \text{ if } \theta > 0 \\ &= \quad 0 \text{ if } \theta = 0 \\ &= -1 \text{ if } \theta < 0. \end{aligned}$$

Consider a particular value of $x = x_0$, and redefine the indices $j = 1, 2, \ldots, N$ so that the values $y_j(x_0)$ are listed in increasing order from left to right across the graph of Figure 2.

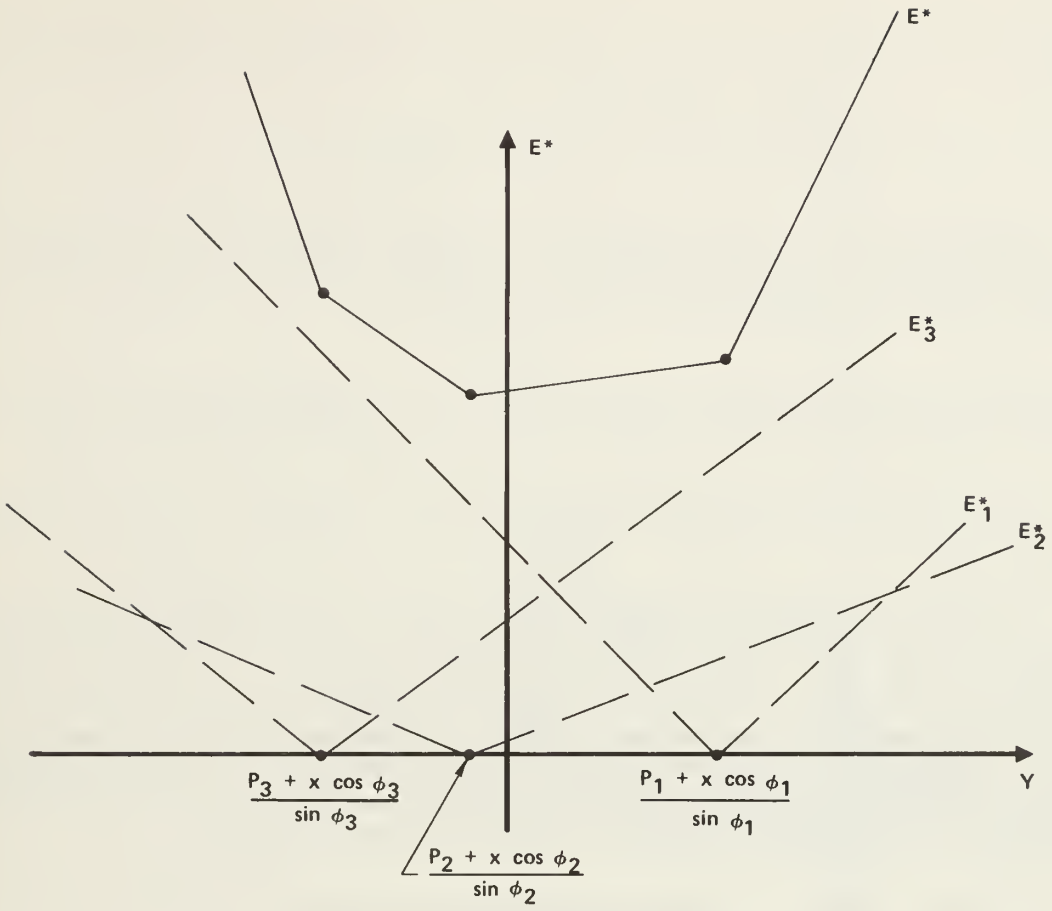

FIGURE 2. Absolute error E^* as a function of y

The value of $y_j(x_0)$, call it $y_r(x_0)$, at which the minimum of $E^*(x_0, y)$ occurs is at the vertex in Equation (5) satisfying the equation

$$\text{sgn}\,(y - y_r(x_0)) = \text{sgn}\left(\frac{\partial E^*(x_0, y)}{\partial y}\right) \tag{8}$$

for all y. For simplicity of notation, we set

$$y_0 = y_r(x_0). \tag{9}$$

Then $y_j(x_0) < y_0$ for $j = 1, 2, \ldots, r-1$, while $y_j(x_0) > y_0$ for $j = r, r+1, \ldots, N$. It then follows from Equations (3) and (5) that

$$\text{sgn}\, E_j^*(x_0, y_0) = \text{sgn}(j - r) \tag{10}$$

so that Equation (6) yields

$$E^*(x_0, y_0) = \sum_{j=1}^{N} \sigma_j^{-1}(\sin \phi_j) E_j^*(x_0, y_0) \,\text{sgn}\,(j - r)$$

$$= K_0 + G_0 x_0$$

where

$$G_0 = \sum_{j=1}^{N} \sigma_j^{-1}(\cos \phi_j) \operatorname{sgn}(j - r) \tag{11}$$

$E^*(x_0, y_0)$ is the minimum possible value of $E^*(x, y)$ when $x = x_0$.

The value of $E^*(x, y)$ can be reduced further by an iterative scheme. Given a value of x_k (beginning with $k = 0$, perhaps taking the least-squares x-value X^μ as x_0), we obtain $y_k = y_r(x_k)$ by the process leading to Equation (9). The choice of y_k defines a value of $r = r_k$ and we then calculate G_k using Equation (11) with r replaced by r_k.

If $G_k > 0$, we choose $x_{k+1} < x_k$, and, if $G_k < 0$, we choose $x_{k+1} > x_k$. The value x_{k+1} should then be taken as small as possible (if $G_{k+1} > 0$) or as large as possible (if $G_{k+1} < 0$) without upsetting the order of the values $y_j(x_{k+1})$, and so without changing G_{k+1} further.

The iteration ends when two successive points (x_k, y_k) and (x_{k+1}, y_{k+1}) are sufficiently close to each other, namely until $|x_k - x_{k+1}| + |y_k - y_{k+1}|$ is sufficiently small.

Equation (2) graphs into a convex polyhedron in the three-dimensional space of coordinates (x, y, E^*) and therefore the iteration process must converge. Some diagrams of this are presented by Karst [6]. As Karst shows, the minimum value of $E^*(x, y)$ will occur either at a unique point (X^*, Y^*), at all points of a unique line segment in the (x, y)-plane, or at all points of a unique simply-connected planar region. In all practical situations, the minimum will occur at a unique point. Special situations, however, can be contrived to result in solution line segments or planar regions. One example of this will be presented in the next section, although the likelihood of such special geometries occurring in practice is negligible.

One more comment: if there are N direction lines, there will be $\binom{N}{2} = \frac{1}{2} N(N-1)$ intersection points ("two-line fixes"). In all practical situations, the unique minimum point (X^*, Y^*) will be one of the intersection points. The reason for this is that, given the result (x_k, y_k) of the kth iteration, the method of choosing y_k guarantees that (x_k, y_k) lies on one of the lines; then we always move x_k (in the direction indicated by the sign of G_k) until an intersection is reached. The fact that the solution (X^*, Y^*) must be an intersection point provides an upper bound on the time spent in the iteration process. In fact, a "brute force" approach could be used to find (X^*, Y^*) by calculating $E^*(x, y)$ for each point of intersection (x, y) and then identifying that intersection for which $E^*(x, y)$ is the smallest.

If N is large, the brute force procedure may be more time-consuming than the minimization approach. However, it may possibly be more easily adapted to continual updating of the least-absolute-deviations fix in situations where the position lines are reported sequentially. A large jump in the location of the least-absolute-deviations fix after a number of sequential steps may indicate either the presence of a section object or movement on the part of the original one.

3. AN EXAMPLE

In Figure 3, we have graphed a set of particularly exceptional, but very illustrative, direction lines indicated by the (ϕ, p) ordered pairs (45°, 3), (45°, 6), (135°, 3), and (135°, 6). All

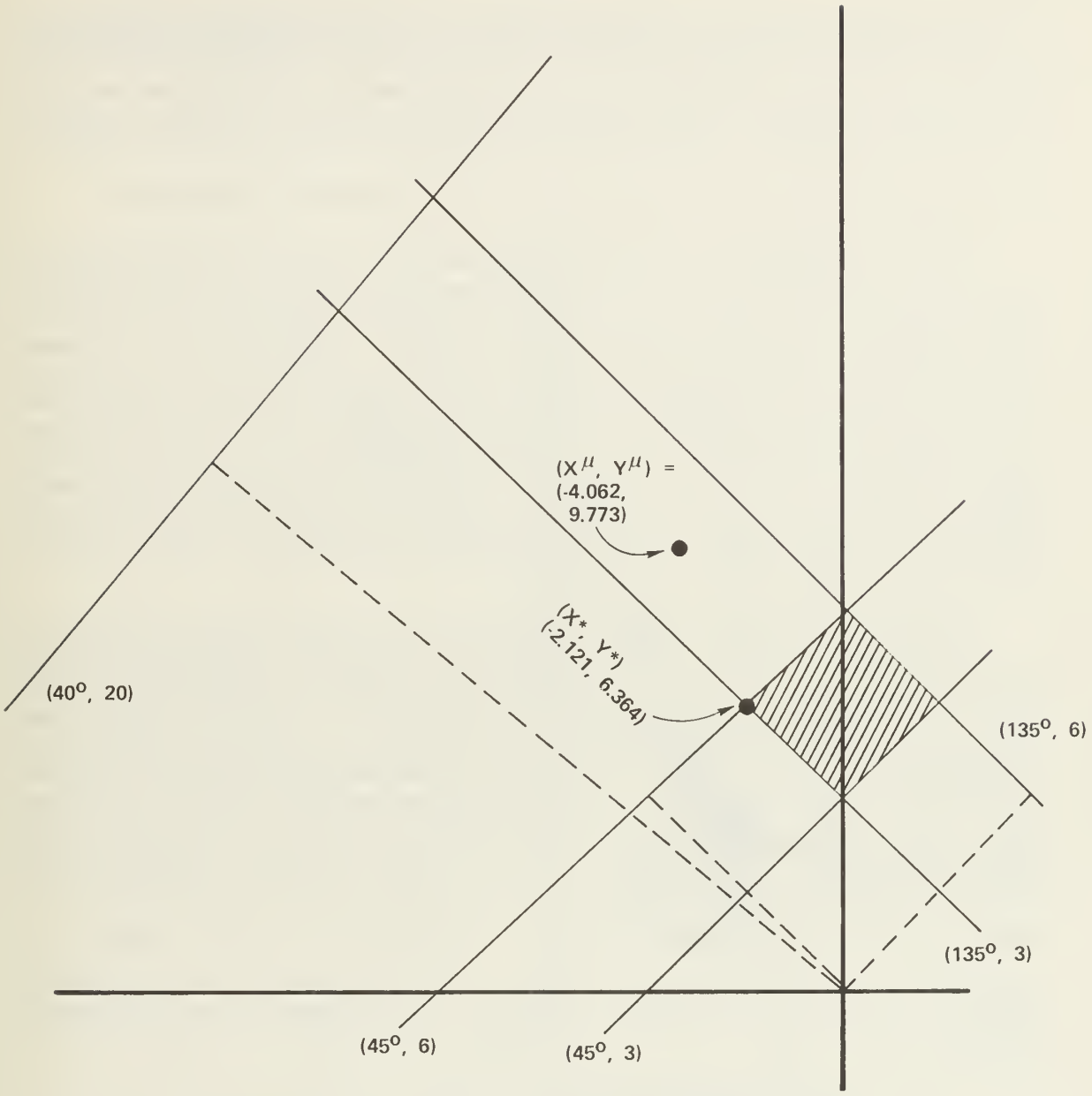

FIGURE 3. Four exceptional direction lines (with a fifth line appended)

σ_j's equal 1. Every point (x, y) inside and on the boundary of the shaded square qualifies as a least-absolute deviation solution (X^*, Y^*) with $E^*(X^*, Y^*) = 6$. The least-squares solution is the point $X^\mu = 0$, $Y^\mu = 6.36396$ at the center of the square, and $E^\mu(X^\mu, Y^\mu) = 9$. Suppose we append a fifth line, with $(\phi, p) = (40°, 20)$, to the set of direction lines in Figure 3. Then the weight produced by this new line pulls the least-squares fix to the point $X^\mu = -4.06242$, $Y^\mu = 9.77273$ outside of the square, while the least-absolute-deviations fix $X^* = -2.12132$, $Y^* = 6.36396$ remains at the far left boundary of the square. It would be reasonable to assume that the line (40°, 20) is an outlier with respect to the object we are trying to locate; the least-absolute-deviations fix is not pulled off course by the presence of the outlier, but the least-squares fix is.

The distinction between the two fixes becomes much more spectacular if the fifth direction line is $(\phi, p) = (30°, 100)$. In this case there is virtually no question at all that the line must be associated with a different object. Yet the least-squares fix is pulled all the way to $X^{\mu} = -27.949$, $Y^{\mu} = 22.5003$. The least-absolute-deviation fix, however, automatically eliminates the effect of the outlier and remains at the corner of the square formed by the first four lines.

4. CONFIDENCE REGIONS FOR THE LEAST-ABSOLUTE-DEVIATIONS FIX

Under the assumption that the direction lines and their associated parallel displacement errors are independent and the errors have zero median (i.e., the true position is as likely to lie on one side of the direction line as on the other), Daniels [1] presented a method of constructing distribution-free confidence regions for the true position. On page 192 of [1] it is shown that, if there are n direction lines, the probability is $P_n = 1 - n2^{1-n}$ that the true position lies within the largest closed polygon having as sides segments of the lines and as vertices pairwise intersections of the lines. Taking $n = 3$, for example, the probability is 0.25 that the true position lies within the triangle formed by a set of three direction lines. Taking $n = 50$, the probability is only 8.88×10^{-14} that the true position lies outside the extremely large polygon formed by the 50 direction lines. For $n = 3$, the region is small but the confidence level is too low, while for $n = 50$, the confidence level is high but the region is too large. What we need is some compromise between these two extremes.

For purpose of discussion, suppose we aim for a 95% confidence region containing the true position. First we can rank all the direction lines according to their distance from the median estimated position (X^*, Y^*). The largest closed polygon formed by the nearest nine direction lines includes the true position with probability $P_9 = 1 - (9)(2^{-8}) = .964844$. Now we have to deal with the remaining direction lines. Making the further assumption that the parallel displacement errors are normally distributed random variables having mean 0 and variance σ_j^2, the probability is $2\Phi(d_j/\sigma_j) - 1$ that the jth direction line is too far from the point (X^*, Y^*) to be considered as an estimate of the true position. Here d_j is the distance of the jth direction line from (X^*, Y^*), and $\Phi(u) = (2\pi)^{-1/2} \int_{-\infty}^{u} \exp(-w^2/2)\,dw$. Therefore, if the n independent direction lines are ranked from 1 to n in order of their distance from (X^*, Y^*), then the probability is $\prod_{j=10}^{n} \{2\Phi(d_j/\sigma_j) - 1\}$ that all direction lines except the nearest nine are not involved in any way with the true position. It therefore follows that the largest polygon formed by the k direction lines nearest to the point (X^*, Y^*) contains the true position with probability

$$(1 - k2^{1-k}) \prod_{j=k}^{n} \{2\Phi(d_j/\sigma_j) - 1\}.$$

As an indication of how this confidence level formula operates, consider a set of $n = 100$ direction lines. Suppose the first nine direction lines have normalized distances d_j/σ_j less than 3.25, the 10th to 14th lines have d_j/σ_j between 3.25 and 3.75, the 15th to the 30th have d_j/σ_j between 3.75 and 4.0, and the remaining 70 lines have d_j/σ_j in excess of 4.0. In this not unlikely scenario, the probability that the largest polygon formed by the nearest nine lines contains the true position is at least

$$(.964844)\ (.9988)^5\ (.9998)^{16}\ (.99994)^{70} = .951997,$$

i.e., this polygon is a 95.2% confidence region for the true position.

Consider the example of the 20 direction lines graphed in Figure 4. The parameters of these lines are listed in Table 1.

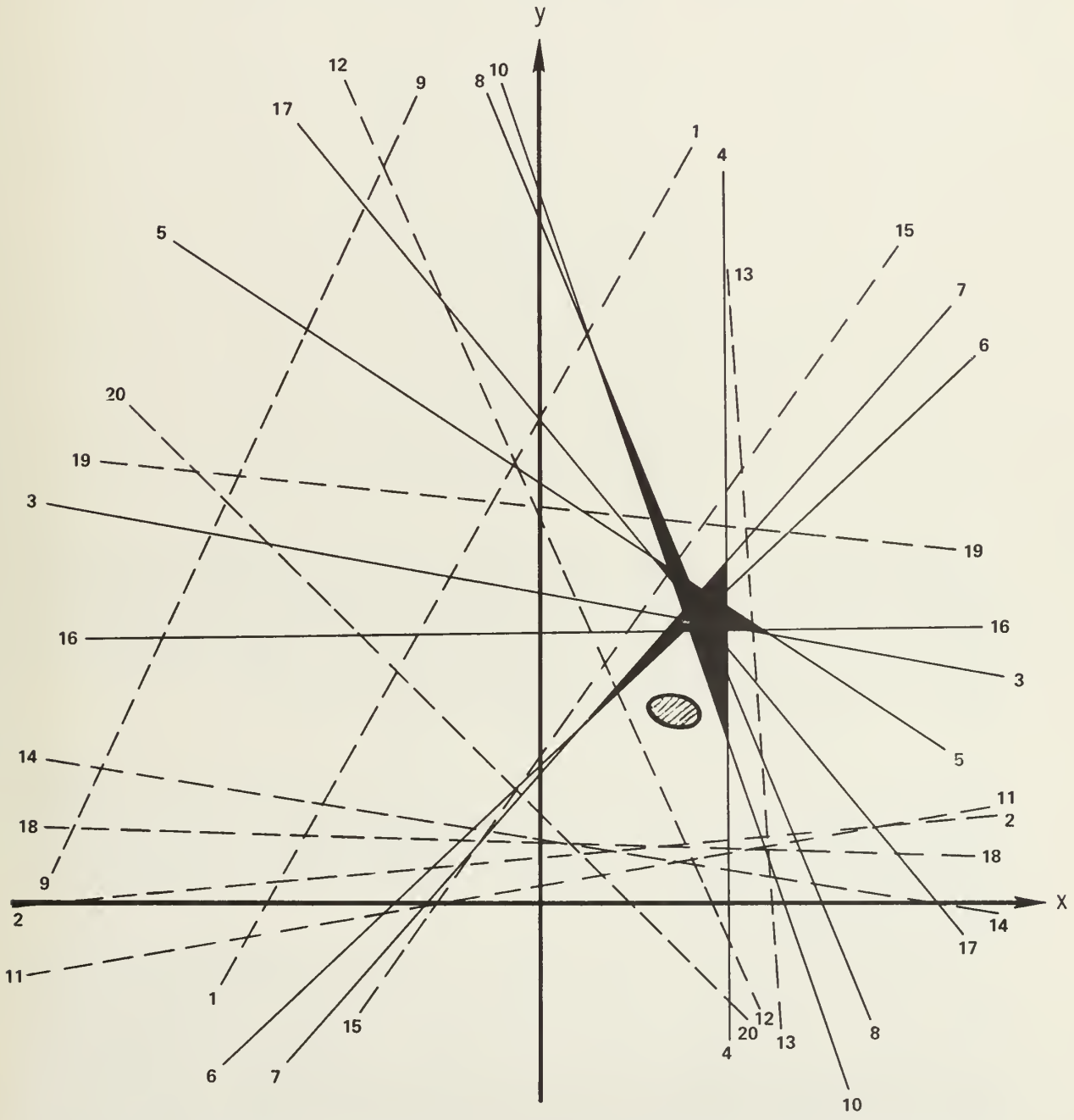

FIGURE 4. Confidence regions in 20 lines example (96.23% confidence polygon and 96.23% confidence Ellipse)

TABLE 1 — *Parameters of the 20 Direction Lines*

Line	ϕ_j	p_j	σ_j
1	30	16	2
2	85	3	1
3	99	20	3
4	179	12	2
5	123	23	1
6	46	7	3
7	40	6	2
8	156	17	2
9	24	30	3
10	160	15	3
11	80	1	1
12	155	10	1
13	174	15	1
14	99	4	2
15	34	6	1
16	89	18	1
17	140	20	1
18	91	2	2
19	95	26	2
20	135	2	3

The least-absolute-deviations median position is at $X^* = 9.35079$, $Y^* = 18.166$. Arranged in order of nearness to the median position, the lines are listed again in Table 2. If we form the polygonal region bounded by or including the nine lines 16, 10, 6, 3, 8, 17, 7, 4, and 5, we note from Table 2 that, of the remaining 11 lines, 2 have d_j/σ_j in excess of 3.54 and the other 9 have d_j/σ_j in excess of 3.80. Because $2\Phi(3.54) - 1 = .9996$ and $2\Phi(3.80) - 1 = .9998$, the polygonal confidence region illustrated in Figure 5 is at least a

$$(.9648)\ (.9996)^2\ (.9998)^9 = .9623 = 96.23\%$$

confidence region for the true (median) position. By way of comparison, the least-squares mean position is at the point $X^\mu = 8.68216$, $Y^\mu = 13.0144$. The standard error of the estimate is 10.8974, and the distance between the mean and median positions is 5.19479. The 96.23% confidence ellipse centered at the mean position is illustrated in Figure 4 and has semimajor axis of length 1.2413, semiminor axis of length 1.0410, and angle of axis rotation of 3.537 degrees in the counterclockwise direction.

Regarding the situation illustrated in Figure 4, a number of interesting points may be raised. Note that none of the 20 position lines passes through (or even very near) the least-squares ellipse. The polygon containing the least-absolute-deviations fix, however, is located at the confluence of several position lines. In this example, even the 99.999% confidence ellipse centered at the least-squares fix would have no position lines passing through it. (The semimajor axis of the 99.999% ellipse is 2.33, but the mean is at distances 2.39 from line 10, 3.37 from line 12, and 3.67 from line 6.) What seems to be happening here is that the ellipse is occupying a balancing position between the confidence polygon above it and the second, less concentrated, confluence of lines below it. Remember that the least-squares fix has the property that

TABLE 2 — Direction Lines Arranged in Order of Nearness to Median Position

Line	d_j	d_j/σ_j	Daniels Confidence Level of Region Formed By this and Nearer Lines
16	0.0	0.0	0.0
10	0.0	0.0	0.0
6	.428	.143	.25
3	.595	.198	.5
8	1.069	.534	.6875
17	1.160	1.160	.8125
7	1.486	.743	.8906
4	2.334	1.167	.9375
5	2.672	2.672	.9648
15	3.594	3.594	.9805
13	3.802	3.802	.9893
12	6.152	6.152	.9941
19	7.088	3.544	.9968
2	14.282	14.282	.9983
1	15.015	7.508	.9991
11	15.266	15.266	.9995
14	15.405	7.703	.9997
18	16.326	8.163	.9999
20	17.457	5.819	.99993
9	31.154	10.385	.99996

the vector sum of its distances from *all* position lines is 0. If line 9 were located very far to the left, therefore, the least-squares ellipse would be drawn across the y-axis. The least-absolute-deviations polygon, however, would remain unchanged by any movement of the outlying lines away from the center. In summary, then, the least-squares ellipse is likely to be located anywhere (not necessarily in the vicinity of any object), if there are several outliers among the set of position lines, while theleast-absolute-deviations polygon will span the centermost concentration of position line intersections.

5. ACKNOWLEDGMENTS

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He would also like to thank the referee for outlining a less pedestrain approach to the minimization process than that presented in an earlier draft of the paper and for offering other valuable suggestions for improvement.

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TIME MINIMIZING FLOWS IN DIRECTED NETWORKS

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ABSTRACT

An important class of network flow problems is that class for which the objective is to minimize the cost of the most expensive unit of flow while obtaining a desired total flow through the network. Two special cases of this problem have been solved, namely, the bottleneck assignment problem and time-minimizing transportation problem. This paper addresses the more general case which we shall refer to as the time-minimizing network flow problem. Associated with each arc is an arc capacity (static) and a transferral time. The objective is to find a maximal flow for which the length (in time) of the longest path carrying flow is minimized. The character of the problem is discussed and a solution algorithm is presented.

1. INTRODUCTION

A natural extension of the bottleneck assignment problem [7] and the time-minimizing transportation problem [8] is the time-minimizing network flow problem. The time-minimizing network flow problem is one of finding, among all maximal flows in a network, the one which minimizes the longest time on any path carrying flow.

We can develop a formal definition of the time minimizing network flow problem as follows. Let x_j denote the flow on path j, b_i the capacity of arc i, a_i the time (length) of arc i, and P the arc - path incidence matrix ($p_{ij} = 1$, if arc i is in path j and 0 otherwise). Then the length, l_j, of a path in the network is defined as

$$l_j = a\ P_j$$

where P_j is the arc incidence vector of path j.

Let $x = [x_j]$ be a vector of path flows, $B(x)$ the set of basic paths, and let v_{max} be the maximal flow in G; then the time-minimizing network flow problem is*

*Note that this formulation will tend to drive a longest path out of a basis even if it is in at a zero level.

PROBLEM 1:

(1) Minimize maximum $\{a\ P_j\}$

$\{x\}$ $P_j \in B(x)$

(2) Subject to $\sum_j x_j \geqslant v_{max}$

(3) $\sum_j p_{ij}\, x_j \leqslant b_i,\ i = 1, \ldots, m$

(4) $x \geqslant 0.$

As is shown by the example in Figure 1 and Table 1, Problem I need not have an all-integer optimum solution. The value of the objective (1) for this problem is $L = 25$, where L is the length of the longest path carrying flow. Since the flows are fractional, we would not expect that there exists an efficient algorithm (like that for maximal flows) for finding time minimizing flows in G.

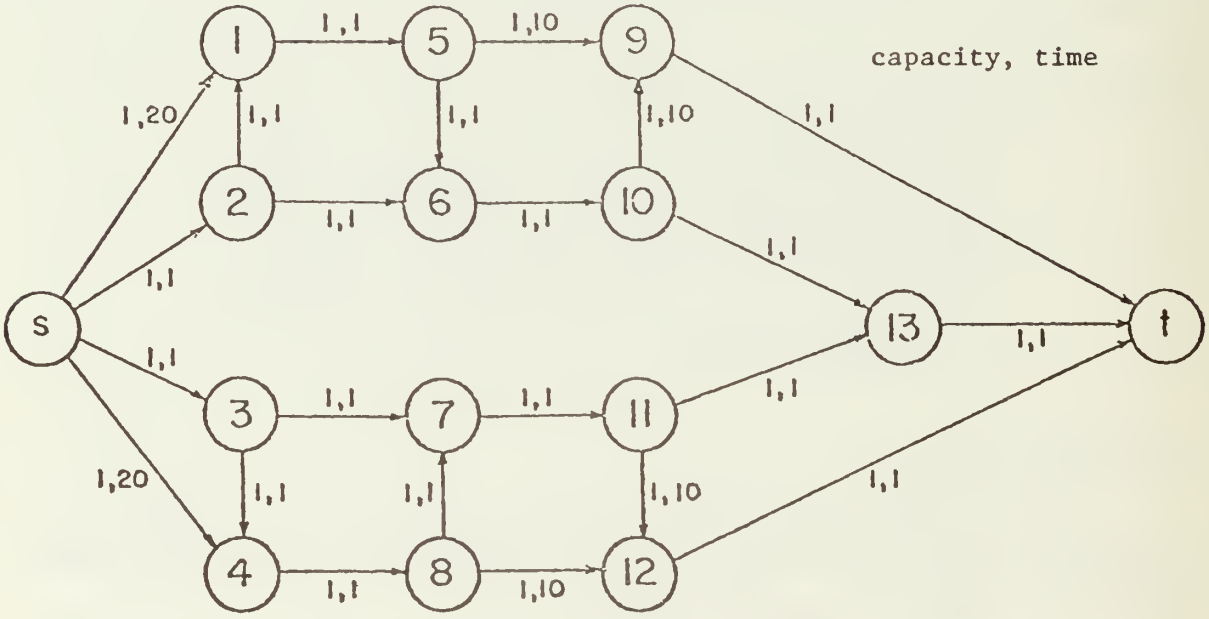

FIGURE 1. Network with fractional time-minimizing flow

TABLE 1 — *(Unique) Time Minimizing Flow Solution for Figure 1*

Path (Nodes)	Length	Flow
s,1,5,6,10,13,t	25	1/2
s,2,1,5,9,t	14	1/2
s,2,6,10,9,t	14	1/2
s,4,8,7,11,13,t	25	1/2
s,3,4,8,12,t	14	1/2
s,3,7,11,12,t	14	1/2

The algorithm proposed herein for solving the time minimizing network flow problem is based on the arc path formulation of the maximum flow problem. The algorithm proceeds by iteratively removing the flow from the longest path(s) in the current basic feasible solution, to Problem I. The steps of the algorithm will be developed along with proofs necessary to establish its validity. Finally, an example is presented.

2. ALGORITHM DEVELOPMENT

Suppose that we have an optimal solution to the maximal flow problem in arc-path form and have enumerated all paths in the network from the source s to the sink t. (Employing column generation, it is not actually necessary to explicitly enumerate all paths.) Let L be the length of the longest path carrying flow in this solution.

To determine whether there exists a solution to the maximum flow problem with:

$$L' \leqslant L - 1,$$

we solve the following problem:

PROBLEM 2:

$$\text{Min } Z = \sum_j \delta_j x_j \tag{5}$$

$$\text{Subject to } Px \leqslant b \tag{6}$$

$$\sum_j x_j \geqslant v_{\max} \tag{7}$$

$$x \geqslant 0. \tag{8}$$

where δ_j is defined as follows ($R >> 0$ to be specified later):

$$\delta_j = \begin{cases} R, & \text{if } j \text{ nonbasic and } l_j \geqslant L \\ 1, & \text{if } j \text{ basic and } l_j \geqslant L \\ 0, & \text{if } l_j < 1 \end{cases}$$

If the optimal solution to Problem 2 has $Z^* = 0$, then this solution has a (min-max path) value of

$$L' \leqslant L - 1,$$

and hence is a better solution to the time-minimizing flow problem. If Z^* is positive, then the previous solution was optimal and the time-minimizing flow has value L.

Associating a Lagrange multiplier, $p > 0$, with constraint (6), we can reformulate Problem 2 as follows:

PROBLEM 3:

$$\min Z = \sum_j \delta_j x_j - p \sum_j x_j = \sum_j (\delta_j - p)\, x_j \tag{9}$$

$$\text{Subject to } Px \leqslant b \tag{10}$$

$$x \geqslant 0. \tag{11}$$

It should be clear that, for p sufficiently large, any optimal solution to Problem 3 will have $\sum_i x_i = v_{max}$, and thus constraint (6) will be satisfied. Hence, by redefining δ_j each time L is reduced we may concentrate on Problem 3 in order to obtain the solution to Problem 1.

Since constraints (10) and (11) are exactly the feasible set of the maximal flow problem (in arc-path form), we may initiate Problem 3 with any maximal flow. (We shall indicate how to find a maximal flow in arc path form later). Assume the current basic feasible solution has been ordered so that those paths with $l_j = L$ are first, those with $l_j < L$ are next, followed by any slack variables. Let $I = \{j | j \text{ basic}, \ l_j = L\}$ and $J = \{j | j \text{ basic}, \ l_j < L\}$. Then $c_B = ((1 - p): -p: 0)$ since $\delta_j = 1$ for $i \in I$ and $\delta_j = 0$ for $i \in J$. Finally, if we denote the dual vector for Problem 3 by ρ then

$$\rho = c_B B^{-1} = ((1 - p): -p: 0)B^{-1} = (1 - p) \sum_{i \in I} B_i^{-1} - p \sum_{i \in J} B_i^{-1} \tag{12}$$

where B_i^{-1} is the ith row of B^{-1}. Thus, ρ is easily computable from the information associated with the current basic solution.

Note that there is another way to compute ρ depending on what information is available. From (12) we can obtain

$$\rho = \sum_{i \in I} B_i^{-1} - p \sum_{i \in I \cup J} B_i^{-1} = \sum_{i \in I} B_i^{-1} - p \ \pi$$

where π is the dual vector for the corresponding maximal flow problem. Thus, we see that ρ is an adjustment of π.

We shall assume that $\rho_k \leqslant 0$, for if such is not the case, we simply enter the slack variable s_k into the basis and obtain $\rho_k = 0$. We shall also assume that $\pi_k \geqslant 0$. For, if not, then $z_k - c_k = \rho_k = \left(\sum_{i \in I} B_i^{-1}\right)_k - p\pi_k > 0$ since $p >> 0$; and thus we may enter stack variable s_k into the basis.

To identify a candidate vector to enter the basis for Problem 3, we make the following observations. First, no (nonbasic) path with $l_j \geqslant L$ is a candidate since $\delta_j = R >> 0$ will produce $z_j - c_j = \rho \ P_j - (R - p) \leqslant 0$. In fact, since $\rho \ P_j \leqslant 0$ we now see that it suffices to choose $R = p$. Thus, we may focus on (simple) paths P_j which satisfy $\rho \ P_j - (-p) > 0$ (i.e. $(-\rho) \ P_j < p$) and $aP_j \leqslant L - 1$.

If one exists, we can locate a candidate path satisfying the aforementioned conditions by solving

$$\begin{aligned} &\text{minimize } (-\rho) \ P_j \\ &\text{Subject to } aP_j \leqslant L - 1 \\ &\qquad\qquad j \ a \text{ path.} \end{aligned}$$

This is a constrained shortest path problem, for which a number of solution algorithms [1, 10, 12] exist.

3. ALGORITHMIC DETAILS

Figure 2 presents a flow chart of the algorithm developed in the previous section. Operations ④ and ⑥ are simplex pivots. Operations ② and ③ are simple computations. Operation 5 is somewhat more involved, but as indicated previously several algorithms do exist for the solution of this subproblem.

Operation ① appears to be quite involved. However, a very simple method exists for developing an initial basis and its inverse. Employ any labeling method (e.g., see [3]) to find a maximal (node-arc) flow. Establish a path decomposition of this flow as follows:

0. Set $k = 1$

1. Label along arcs with positive flow to obtain a path, k, from source to sink.

2. Put as much flow on path k as possible, thereby reducing some arc flow (on the path) to zero. Reorder the arcs so that this becomes arc k. Establish P_k.

3. Set $k = k + 1$ and repeat steps 1 and 2 until all flow has been decomposed. Go to step 4 with the last value $k = q$.

4. Fill out the remaining basis members with slack variables for each of the remaining arcs. Then

$$B = \left[P_1, P_2, \ldots, P_q \,\middle|\, \frac{0}{I_{m-q}} \right]$$

Note that B is upper triangular since P_k has zeros in elements $1, 2, \ldots, k-1$ and a 1 in element k. Thus,

$$B^{-1} = I - (B - I).$$

4. EXAMPLE

We shall use the network of Figure 3 to illustrate the calculations for solving the constrained shortest path problem. The circled number adjacent to each arc is the flow on that arc with respect to maximum flow on the network. All arc capacities are one and the arc lengths are given on each arc.

The maximal flow (minimal cost) solution is given by one unit of flow on $P_1 = \{1, 4, 8, 10\}$ and one unit of flow on $P_2 = \{2, 6, 11\}$. We consider P_1 and P_2 to be basic with respect to arcs one and two, respectively. The remainder of the basis consists of the appropriate slack variables.

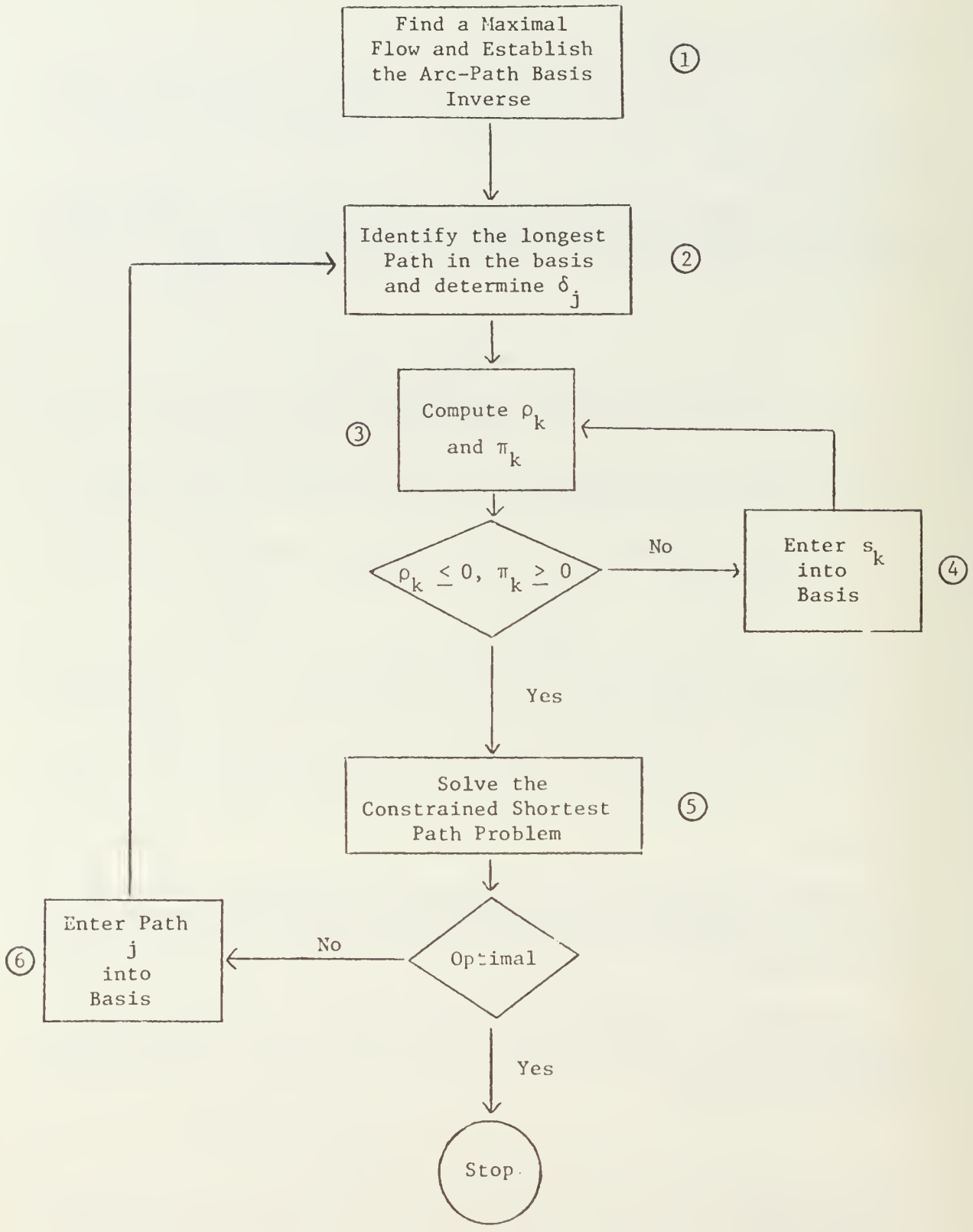

FIGURE 2. Flow chart of the time-minimizing network flow algorithm

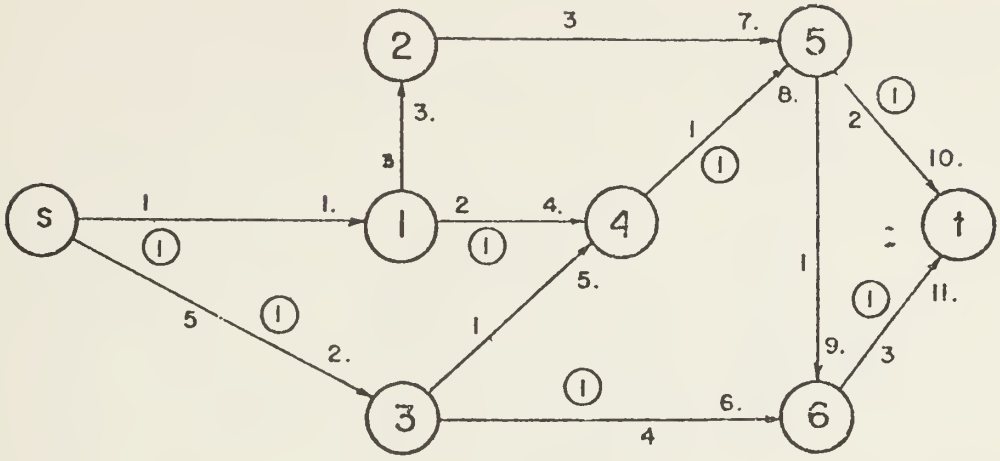

FIGURE 3. Initial path flow

For this maximal flow solution:

$$B = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0 \\ 1&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0 \\ 0&1&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0 \\ 1&0&0&0&0&0&0&0&0&1&0 \\ 0&1&0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

and

$$B^{-1} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0 \\ -1&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0 \\ 0&-1&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0 \\ -1&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0 \\ -1&0&0&0&0&0&0&0&0&1&0 \\ 0&-1&0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

After two iteractions we still have $x_1 = 1$, and $x_2 = 1$, but we also have $P_3 = \{1, 3, 7, 10\}$ and $P_4 = \{2, 5, 8, 10\}$ in the basis (for slacks 8 and 10, respectively) with $x_3 = x_4 = 0$.

At this point we have:

$$L = l_2 = 12,$$

$$L - 1 = 11,$$

and

$$B^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
-1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}$$

Also, we have

$$\pi = (1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

Since the second basic path is the longest, we have

$$B_2^{-1} = (1, 1, 0, 0, 0, 0, 0, 0, 0, -1, 0).$$

If we let $p = 4$, we have

$$p\pi - B_2^{-1} = (3, 3, 0, 0, 0, 0, 0, 0, 0, 1, 0).$$

Thus, we wish to

minimize $(3, 3, 0, 0, 0, 0, 0, 0, 1, 0)\ P_j$

Subject to: $(1, 5, 3, 2, 1, 4, 3, 1, 1, 2, 3)\ P_j \leqslant 11$

P_j is a path from s to t.

The optimal solution to this constrained shortest path problem is

$$P_5 = \{2, 5, 8, 9, 11\}.$$

If we bring this path into the basis (replacing P_2) we obtain a solution:

$$x_5 = x_3 = 1$$

$$x_1 = x_4 = 0.$$

The remaining basic variables are slack variables.

In Figure 4 the arc flows corresponding to the current solution are shown by circled numbers. For this solution

$$L = l_5 = 11$$

$$L - 1 = 10,$$

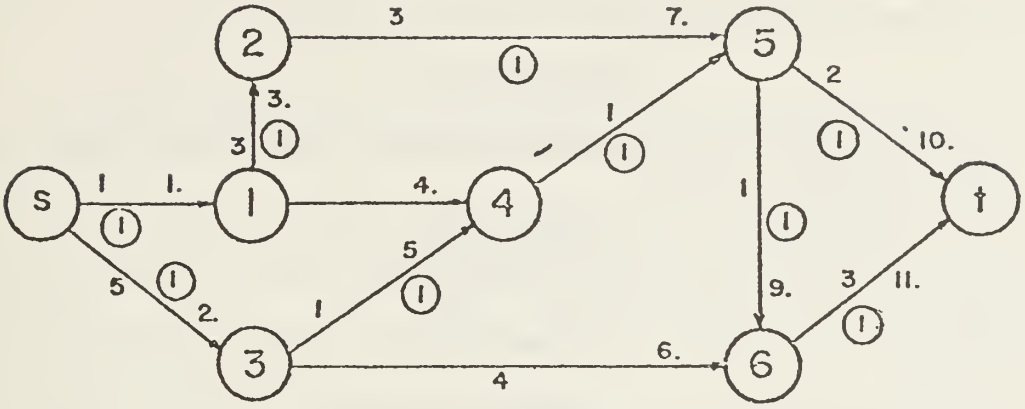

FIGURE 4. New path flow

and

$$B^{-1} = \begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
-1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}$$

The simplex multipliers are

$$\pi = (1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

If we let $p = 4$, we have

$$\underline{d} = 4\pi - B_2^{-1} = (3, 3, 0, 0, 0, 0, 0, 0, 0, 1, 0).$$

The optimal solution to the corresponding constrained shortest path problem is $P_6 = \{1, 4, 8, 9, 11\}$. After pivoting we have

$$x_5 = x_3 = 1$$

$$x_6 = x_4 = 0.$$

The longest path in the basis remains P_5. We have

$$L - 1 = 10$$

$$B_2^{-1} = (1, 2, 0, 0, 0, 0, 0, -1, 0, -1, 0)$$

$$\pi = (1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$4\pi - B_2^{-1} = (3, 2, 0, 0, 0, 0, 0, 1, 0, 1, 0).$$

With these arc costs, the optimal constrained shortest path is P_3 with reduced cost equal to 0. Thus, we conclude that the current solution to the problem is optimal.

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TRANSPORTATION PROBLEMS WITH AGGREGATED DESTINATIONS WHEN DEMANDS ARE UNCERTAIN

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ABSTRACT

Transportation problems with uncertain demands are useful applied models themselves, and also they represent in a formal way the problem of estimating demands for use in deterministic models. We consider the effects of using a small, aggregate model of this type in place of a larger, more detailed one. Formulation of the aggregate objective function turns out to depend on how one chooses to use ("disaggregate") the solution; several alternative methods are examined. Bounds are derived on the error induced by the approximation, thus facilitating comparison of alternative aggregations. We also consider the problem of estimating demands for an aggregate-level deterministic problem. In a specific sense, it is often *not* the case (as one might expect) that such aggregate demands are easier to estimate than the detailed demands. This is because aggregation and centralization are not the same thing.

INTRODUCTION

This is a study of transportation problems whose destinations represent aggregates of smaller, "original" destinations, and where the demands at the destinations are not known with certainty.

The models analyzed represent uncertainty by means of an additional cost incurred when total shipments to a destination exceed or are exceeded by the true demand. Such additional costs may represent emergency procurement, lost sales, disposal, salvage, and the like. Thus, the models are stochastic linear programs with simple recourse, representable as ordinary nonlinear programs. These have been studied extensively by Williams [10, 11, 12], among others. We shall refer to such a "transportation problem with uncertain demands" as a TPUD.

This paper has two purposes, reflecting two distinct aims of prior studies of the TPUD. First, a TPUD can itself be used in practice, like other models, to help solve a given decision problem (e.g., [4]). From this viewpoint our analysis follows the same lines as earlier work on aggregation of deterministic network-type problems (Kuhn and Cullen [8], Geoffrion [5, 6], Evans [2, 3], and Zipkin [13, 16]):

Suppose a large, original TPUD is given (Section 2), and we approximate it by aggregating groups of destinations. Upon solving the aggregate problem, suppose we were to disaggregate its solution, obtaining a feasible solution to the original problem. The cost of such a disaggregated solution would provide a natural specification of the objective function of the aggregate

problem, as well as a natural criterion for the error resulting from substituting the aggregate for the original problem. In Section 3 we consider two distinct approaches to disaggregation, "fixed-weight" and "optimal" disaggregation. In every case, the aggregate problem is itself a TPUD, and under certain assumptions ((A1) and, for fixed-weight disaggregation, (A2)), the demand distributions in this problem are of reasonably simple form. Under optimal disaggregation the aggregate objective function is derived by an approximation method which has not been used before in this context.

Section 5 presents bounds on this measure of error, which do not require solution of the original TPUD. These may be used, therefore, to help choose among alternative aggregations and to evaluate the solution for a particular aggregation. Analysis of the form of the bounds also suggests qualitatively which factors are important in the design of good aggregations.

The second reason for studying the TPUD is that it represents a formalization of the problem of estimating demands for use in a classical transportation problem (CTP). The objective function of the TPUD is now interpreted as the total expected loss (in the Bayesian sense) as a function of the estimates, where the estimates correspond to total shipments to the respective destinations. (This interpretation is suggested, for example, in [12] for general stochastic linear programs.)

When, as is often the case, the destinations of the CTP represent aggregates, it is natural to use the true distribution of total demand as the basis for (Bayesian) estimation of demand at each destination. The discussion in Section 4, based on examination of the aggregate objective function, suggests that this approach is *incorrect*, *unless* aggregation of the model is accompanied by actual centralization in the system being modeled. The correct distributions have larger variances than those of the true aggregate demands.

Similarly, if demands are estimated by the usual classical techniques, we argue that aggregation *per se* may *not* reduce overall data requirements.

These peculiar results are explained by the difference between aggregation of a model and actual centralization of the distribution system represented by the model. They demonstrate, we believe, that the usual, intuitive view of the meaning of aggregation may in some cases be misleading.

2. THE ORIGINAL PROBLEM

We state the original, detailed TPUD as follows:

$$
\begin{aligned}
z^* = \min\ & \sum_{ij} c_{ij}x_{ij} + \sum_j a_j(y_j) \\
\text{subject to}\ & \sum_j x_{ij} \leqslant s_i,\ i \in I \qquad (1) \\
& \sum_i x_{ij} - y_j = 0,\ j \in J \\
& x_{ij} \geqslant 0,\ i \in I,\ j \in J,
\end{aligned}
$$

where

$$
a_j(y_j) = \gamma_j \int_{y_j}^{\infty} (\xi - y_j)\, dF_j(\xi) - \delta_j \int_{-\infty}^{y_j} (y_j - \xi)\, dF_j(\xi).
$$

The notation (mostly following Williams [11]) is defined as follows:

I	=	index set for sources
J	=	index set for destinations
x_{ij}	=	shipment from source i to destination j
y_j	=	total shipment to destination j
c_{ij}	=	unit shipment cost from i to j
s_i	=	supply at source i
F_j	=	cumulative distribution of demand at destination j
γ_j	=	unit cost of undersupply at j
δ_j	=	unit salvage value (or minus oversupply cost) at j
z^*	=	minimal total cost.

The summations are defined over appropriate subsets.

The following assumptions are made to avoid trivialities and to simplify the notation:

a. $\gamma_j \geqslant \min_i\{c_{ij}\} \geqslant \delta_j$ and $\gamma_j > \delta_j$, $j \in J$.

b. $s_i > 0$, $i \in I$.

c. Each F_j has finite, positive mean μ_j. Also, each F_j is sufficiently continuous that a_j can be written in the simpler form,

$$a_j(y_j) = \gamma_j\mu_j - \gamma_j y_j + (\gamma_j - \delta_j)\int_{-\infty}^{y_j} F_j(\xi)\,d\xi. \tag{2}$$

These assumptions are sufficient to ensure that (1) is a convex program with a finite optimum. Let $x^* = (x_{ij}^*)$ and $y^* = (y_j^*)$ denote such an optimum. Also, let $\pi^* = (\pi_i^*)$ and $\rho^* = (\rho_j^*)$ denote an optimal dual solution to (1).

The optimality conditions for problem (1) are as follows:

$$\pi_i^* \leqslant 0 \text{ and } \pi_i^*\Big(\sum_j x_{ij}^* - s_i\Big) = 0,\ i \in I;$$

$$\pi_i^* + \rho_j^* \leqslant c_{ij} \text{ and } x_{ij}^*(\pi_j^* + \rho_j^* - c_{ij}) = 0,\ i \in I,\ j \in J;$$

$$(x^*, y^*) \text{ feasible for (1)};$$

$$y_j^* = \inf\{y_j: y_j \geqslant 0,\ F_j(y_j) \geqslant (\gamma_j - \rho_j^*)/(\gamma_j - \delta_j)\},\ j \in J.$$

To simplify the latter condition we may at times invoke the following assumption:

d. Each F_j is continuous and increasing over its positive range. In this case, the last condition reduces to the following:

$$F_j(y_j^*) = (\gamma_j - \rho_j^*)/(\gamma_j - \delta_j),\ j \in J. \tag{3}$$

Viewing (1) as a problem of Bayesian estimation of demands, the condition (3) suggests a heuristic approach to determining "certainty equivalents" y_j^* [12]: First try to estimate each marginal shipment cost ρ_j^*. Then, use the results with (3) to estimate the y_j^*. The role played by the F_j (interpreted as summarizing beliefs concerning "true" demands) in this "metamodel" of the estimation process will be significant in the discussion in Section 4 below.

3. AGGREGATION AND DISAGGREGATION

Aggregation: By aggregation of destinations we mean the following: Partition the destinations J into subsets T_k, $k \in K$, and replace each subset T_k by the single "aggregate" destination k. No restrictions are placed on the partition. (If the original problem includes only a subset of the arcs $I \times J$, then each $j \in T_k$ must be linked to the same subset of sources, $k \in K$.) Denote the *aggregate problem* as follows (the numbers c_{ik}^A and the functions a_k^A are thus far undefined):

$$z^0 = \min \sum_{ik} c_{ik}^A x_{ik}^A + \sum_k a_k^A(y_k^A)$$

$$\text{subject to } \sum_k x_{ik}^A \leqslant s_i, \quad i \in I \tag{4}$$

$$\sum_i x_{ik}^A - y_k^A = 0, \quad k \in K$$

$$x_{ik}^A \geqslant 0, \quad i \in I, \ k \in K.$$

Let $k(j)$ denote that k for which $j \in T_k$, $j \in J$. The superscript A will be used to distinguish aggregate-level variables and data. Otherwise, the notation is analogous to that of problem (1). Let (x^{A0}, y^{A0}) denote an optimal solution to (4) (assuming one exists), and (π^{A0}, ρ^{A0}), a dual-optimal solution.

Aggregation with Fixed-Weight Disaggregation: Consider the following method of defining the objective function of (4): Choose any positive weights w_j, $j \in J$ with $\sum_{T_k} w_j = 1$, and set

$$c_{ik}^A = \sum_{T_k} w_j c_{ij}, \quad i \in I, \ k \in K$$

$$a_k^A(y_k^A) = \sum_{T_k} a_j(w_j y_k^A), \quad k \in K.$$

(Viewing (1) as a convex-cost network-flow problem, this objective function is a special case of the one proposed in [16].)

This method of aggregation is closely related to *fixed-weight disaggregation*: Given an optimal solution (x_{ik}^{A0}, y_k^{A0}) to problem (4) and weights $w = (w_j)$, define a solution (x^0, y^0) to problem (1) as follows:

$$x_{ij}^0 = w_j x_{ik(j)}^{A0}, \quad i \in I, \ j \in J.$$

$$y_j^0 = w_j y_{k(j)}^{A0}, \quad j \in J.$$

It can be shown [16] that (x^0, y^0) is feasible for (1) and that

$$\sum_{ij} c_{ij} x_{ij}^0 + \sum_j a_j(y_j^0) = \sum_{ik} c_{ik}^A x_{ik}^{A0} + \sum_k a_k^A(y_k^{A0}) = z^0.$$

We may interpret these facts as follows: The ultimate result of solving (4) is the feasible (but perhaps suboptimal) solution (x^0, y^0). Assuming the given weights w are to be used for disaggregation, solution of (4) yields that solution whose disaggregation is of minimal possible cost in problem (1).

Evidently, $z^0 \geqslant z^*$. In Section 5 we shall derive bounds on $z^0 - z^*$.

The definition of the a_k^A above ensures that (4) is a convex program. It is perhaps less obvious that (4) is a TPUD, that is, that a_k^A is of form (2) (except perhaps for a different constant term). For each $k \in K$, define

$$\gamma_k^A = \sum_{T_k} w_j \gamma_j, \quad \delta_k^A = \sum_{T_k} w_j \delta_j.$$

We have

$$a_k^A(y_k^A) = \sum_{T_k} \gamma_j \mu_j - \sum_{T_k} \gamma_j w_j y_k^A + \sum_{T_k} (\gamma_j - \delta_j) \int_{-\infty}^{w_j y_k^A} F_j(\xi)\, d\xi.$$

The first term is a constant, and the second term is simply $-\gamma_k^A y_k^A$. Changing variables, and defining $v_j = w_j(\gamma_j - \delta_j)/(\gamma_k^A - \delta_k^A)$, the third term becomes

$$\begin{aligned}\sum_{T_k} (\gamma_j - \delta_j) \int_{-\infty}^{y_k^A} w_j F_j(w_j \xi)\, d\xi &= \int_{-\infty}^{y_k^A} \sum_{T_k} w_j (\gamma_j - \delta_j) F_j(w_j \xi)\, d\xi \\ &= (\gamma_k^A - \delta_k^A) \int_{-\infty}^{y_k^A} \sum_{T_k} v_j F_j(w_j \xi)\, d\xi \\ &= (\gamma_k^A - \delta_k^A) \int_{-\infty}^{y_k^A} F_k^A(\xi)\, d\xi,\end{aligned}$$

where $F_k^A(\xi) = \sum_{T_k} v_j F_j(w_j \xi)$, for all ξ. Since $\sum_{T_k} v_j = 1$, F_k^A is a cumulative distribution function, so a_k^A is of form (2).

F_k^A satisfies assumption c in Section 2. If the F_j satisfy assumption d, so does F_k^A. It is easy to shown that $\min_i\{c_{ik}^A\} \geqslant \delta_k^A$ and $\gamma_k^A > \delta_k^A$, but it may not be true that $\gamma_k^A \geqslant \min_i\{c_{ik}^A\}$. We simply assume this to be true; if not, no shipment would be made to k.

The functions F_k^A are point mixtures of rescaled versions of the original demand distributions F_j, $j \in T_k$. In general, distributions of this form are not easy to work with. For example, the original F_j may be easy to invert, as required by the optimality conditions (3), but this property may not be inherited by the F_k^A. Unless the F_k^A do have simple forms, one would probably wish to approximate them in practice.

It turns out, however, that under certain conditions the F_k^A *are* tractable. Specifically, suppose the following two assumptions hold:

(A1) *For each $k \in K$ there exists a probability distribution H_k with mean 1 such that $F_j(\xi) = H_k(\xi/\mu_j)$ for all ξ and $j \in T_k$.*

(A2) *Let $\mu_k^A = \sum_{T_k} \mu_j$ and $m_j = \mu_j/\mu_k^A$, $j \in T_k$, $k \in K$. Set the weights $w_j = m_j$, $j \in J$.*

The first assumption states that the demand distributions within each subset are identical, except for a scale factor. Several well-known families of distributions satisfy this condition, for example:

(i) Each F_j is gamma (or each is Weibull, or each is Pareto), with a common shape parameter;

(ii) Each F_j is normal, with a common coefficient of variation.

(A1) thus imposes a not entirely unreasonable model of demands of different sizes for the same item. Assumption (A2) requires chosing weights proportional to the mean demands, which is also reasonable.

Under these conditions we have, for any ξ,

$$F_k^A(\xi) = \sum_{T_k} v_j H_k(m_j \xi/\mu_j)$$
$$= \sum_{T_K} v_j H_k(\xi/\mu_k^A) = H_k(\xi/\mu_k^A).$$

Thus, F_k^A is of the same family of distributions as the F_j, and is just as tractable. The constant term in a_k^A here is $\sum_{T_k} \gamma_j \mu_j = \gamma_k^A \mu_k^A$.

The mean of F_k^A is just μ_k^A, the sum of the individual means. Also, let σ_k^2 denote the variance of H_k. Then, the variance of F_j is $(\mu_j \sigma_k)^2$, $j \in T_k$, and the variance of F_k^A is $(\mu_k^A \sigma_k)^2$. Thus, F_k^A would be the distribution of the sum of the demands at destinations $j \in T_k$, *if* those demands were perfectly correlated. We shall return to this observation in Section 4.

Aggregation with Optimal Disaggregation, First Form: "Optimal" disaggregation has been defined (e.g., [13]) as any method of obtaining a solution to (1) from a solution to (4) which improves on the solution derived from fixed-weight disaggregation, usually by solving by solving one or more relatively simple optimization subproblems. The special form of the TPUD suggests several natural forms of optimal disaggregation.

The first form we shall consider is as follows: Define y^0 as in fixed-weight disaggregation, but obtain x^0 as the solution to the CTP with demands y^0 and costs and supplies as given in (1). Since the value of x in fixed-weight disaggregation is feasible for this CTP, the resulting solution is obviously of no greater (and usually of lesser) cost.

It would be useful to be able to define the objective function of the aggregate problem to reflect the fact that optimal disaggregation would be used to recover a solution to (1). For this form of optimal disaggregation, unfortunately, there appears to be no such definition which is clearly better than the one above. Solution of that aggregate problem, therefore, minimizes an upper bound on the cost of the solution ultimately obtained.

In Section 5 we shall mention a bound on the suboptimality of this (x^0, y^0), due to Williams [12].

Aggregation with Optimal Disaggregation, Second Form: The other form of optimal disaggregation we shall consider focuses on improvement of the shortage/disposal costs, as opposed to the distribution costs. First we describe the method, and then justify it.

If $y^{A0}_{k(j)} = 0$, set $y^0_j = 0$, and $x^0_{ij} = 0$, $i \in I$, as in fixed-weight disaggregation. Otherwise, define

$$\Gamma_j = \Big(\sum_i c_{ij} x^{A0}_{ik(j)}\Big)/y^{A0}_{k(j)},$$

and, for each k, solve the following subproblem, with $y^A_k = y^{A0}_k$:

$$q_k(y^A_k) = \sum_{T_k} \gamma_j \mu_j + \min \sum_{T_k} \Big[-(\gamma_j - \Gamma_j) y_j + (\gamma_j - \delta_j) \int_{-\infty}^{y_j} F_j(\xi)\, d\xi\Big]$$

$$\text{subject to } \sum_{T_k} y_j = y^A_k \tag{5}$$

$$y_j \geqslant 0,\ j \in T_k.$$

Let y^0_j denote the optimal y_j in problem (5) for $k(j)$. Then, set

$$x^0_{ij} = (y^0_j / y^{A0}_{k(j)})\, x^{A0}_{ik(j)},\ i \in I \cdot j \in J. \tag{6}$$

First of all, the resulting solution $(x^0,\ y^0)$ is feasible for (1), since $x^0_{ij} \geqslant 0$ for all i, j, and

$$\sum_j x^0_{ij} = \sum_k x^{A0}_{ik} \leqslant s_i,\ i \in I,$$

$$\sum_i x^0_{ij} = (y^0_j / y^{A0}_{k(j)}) \sum_i x^{A0}_{ik(j)} = (y^0_j / y^{A0}_{k(j)}) y^{A0}_{k(j)} = y^0_j,\ j \in J.$$

Moreover, the distribution cost of $(x^0,\ y^0)$ is

$$\begin{aligned}\sum_{ij} c_{ij} x^0_{ij} &= \sum_{ij} c_{ij} (y^0_j / y^{A0}_{k(j)})\, x^{A0}_{ik(j)} \\ &= \sum_j \left[\left(\sum_i c_{ij} x^{A0}_{ik(j)}\right)/y^{A0}_{k(j)}\right] y^0_j \\ &= \sum_j \Gamma_j y^0_j.\end{aligned}$$

Thus, the objective function in (5) includes the distribution cost of the x^0 ultimately obtained. Finally, the y from fixed-weight disaggregation is feasible for all the problems (5), and if it is used in (6) to define x, the result is just the x of fixed-weight disaggregation. Thus, $(x^0,\ y^0)$ is no more (and usually less) costly than the solution derived by fixed-weight disaggregation.

Zipkin [15] (following earlier work in the references therein) has developed quite efficient methods to solve problems of form (5).

For this form of optimal disaggregation it *is* possible to specify the objective function of (4) so as more nearly to reflect the true cost of the disaggregated solution, namely $\sum_k q_k(y^{A0}_k)$. This requires assuming (A1) and *d*. Zipkin [17] shows that a function of the form of q_k can be well-approximated by an explicit function of the following form:

$$q_k(y^A_k) \approx \sum_{T_k} \gamma_j \mu_j - (\gamma^A_k - \Gamma^A_k) y^A_k + (\gamma^A_k - \delta^A_k) \int_{-\infty}^{y^A_k} F^A_k(\xi)\, d\xi. \tag{7}$$

Here, the terms $(\gamma_k^A - \Gamma_k^A)$ and $(\gamma_k^A - \delta_k^A)$ are constants. Several methods for determining these constants are given in [17], each with its particular advantages. (They are written as differences to facilitate the discussion below, where actual values of γ_k^A and δ_k^A are derived. Note that these are not the same values as those defined for fixed-weight disaggregation.)

To use the approximation (7) to construct an objective function for the aggregate problem, one must specify several quantities. First, since the approximation must be made before solving (4), the Γ_j must be estimated. That is, for each k, one must estimate the proportion of y_k^{A0} which comes from each source i, and use these estimates in place of the true proportions in the definition of Γ_j above. Second, q_k includes distribution costs, and these must be deleted from (7) in order to define a_k^A. For this purpose estimate weights w_j to approximate y_j^0/y_k^{A0}. (For example, given Γ_j, drop the constraints of (5) and solve for each j separately. Then choose w_j proportional to the resulting y_j.) If the estimates of w_j and Γ_j were exact, then the distribution cost of y_j^0, $j \in T_k$, would be

$$\sum_{T_k} \Gamma_j y_j^0 = \left(\sum_{T_k} w_j \Gamma_j\right) y_k^{A0}.$$

Thus, set

$$\Gamma_k^A = \sum_{T_k} w_j \Gamma_j,$$

and use this in conjunction with the constants determined as in [17] to define γ_k^A and δ_k^A. With these values we can define aggregate shortage/disposal costs:

$$a_k^A(y_k^A) = \sum_{T_k} \gamma_j \mu_j - \gamma_k^A y_k^A + (\gamma_k^A - \delta_k^A) \int_{-\infty}^{y_k^A} F_k^A(\xi)\, d\xi. \tag{8}$$

Finally, use the same w_j to construct aggregate distribution costs, as in fixed-weight disaggregation, i.e.,

$$c_{ik}^A = \sum_{T_k} w_j c_{ij}, \quad i \in I, \ k \in K.$$

Several remarks on this approximation scheme are in order. First, the definition (8) implies that the aggregate problem is a TPUD with the same distributions of aggregate demand F_k^A as result from fixed-weight disaggregation; only the constants differ. Second, as shown in [17], even though (7) is derived by fitting certain properties of q_k at one or two given values of y_k^A, the approximation is close globally, in a certain sense. Third, the approximation is nearly exact if each of the sets $\{\gamma_j - \Gamma_j : j \in T_k\}$ and $\{\gamma_j - \delta_j : j \in T_k\}$ has nearly identical elements. Fourth, if the assumption that all $y_j^0 > 0$ should happen to be false, the approximation still has some desirable properties, though they are weaker than the above. Fifth, the approximation method (but not the results above for fixed-weight disaggregation) can be readily extended to a case more general than (A1) which includes all normal distributions. Specifically, assume $F_j(\xi) = H_k[(\xi - \mu_j)/\sigma_j]$, for positive constants μ_j and σ_j, $j \in T_k$. (If we choose H_k with zero mean and unit variance, of course, μ_j is the mean of F_j, and σ_j is its standard deviation.) Then, (7) still provides a good approximation of q_k, in the sense of [17], where F_k^A is redefined as $F_k^A(\xi) = H_k[(\xi - \mu_k^A)/\sigma_k^A]$, and $\sigma_k^A = \sum_{T_j} \sigma_j$.

The approximation (7) does not necessarily result in an upper bound on q_k, so no particular relationship between z^0 and z^* can be assumed.

4. ESTIMATING AGGREGATE DEMANDS

Define G_k to be the actual distribution of the sum of the demands at destinations $j \in T_k$, $k \in K$.

Suppose one formulated a TPUD from scratch at the aggregate level, that is, with destinations $k \in K$. It would be natural to define the procurement/disposal cost functions a_k^A in terms of the distributions G_k. Similarly, if one were trying to estimate demands for an aggregate-level CTP, one would normally seek to estimate the true total demands. From a Bayesian viewpoint, such estimation would be based on the functions G_k.

In either interpretation, smaller correlations among the individual demands would be reflected in a smaller variance for G_k. This fact expresses the Bayesian version of the standard classical-statistical (and intuitive) result that an aggregate index is "easier" to estimate than its individual components. (The F_j and the G_k can be interpreted as *a posteriori* distributions based on the same data set.) Statements in the literature (e.g., [7], [14]) implying that aggregation reduces data requirements (and, we suspect, much current practice) reflect the same idea.

In the analysis of the previous section, however, it was shown that F_k^A is the appropriate distribution to use at the aggregate level under both fixed-weight and optimal disaggregation, while G_k is identical to F_k^A only if demands are in fact perfectly correlated. Otherwise, while they have the same mean, F_k^A has a larger variance. That analysis thus appears to conflict with the observations above. As we shall now discuss, *this conflict reveals the limitations of our analysis, but it also suggests that the "natural" use of G_k at the aggregate level is often inappropriate.*

The mechanics of aggregation and disaggregation bear a strong resemblance to a partially centralized distribution system. In the latter, typically, y_k^{A0} would be sent first to a central warehouse, and only then divided up among the individual destinations in T_k. The most important advantage of such a system, of course, is that some of the distribution decisions can be postponed until the individual demands are partially realized; or, in other words, the uncertainties of the individual demands can be pooled and their costs thus reduced. See, for example, Eppen's recent study of the effects of centralization on newsboy problems [1].

The point is that no such centralization occurs in the scenarios analyzed above. The results of aggregation and disaggregation (even optimal disaggregation) pertain to a situation modeled by problem (1); in particular, all distribution decisions must be implemented before demand is realized. Aggregation is an operation performed *on a model*, and unless it is accompanied by actual centralization *in practice*, the advantages of centralization mentioned above are not realized. In short, aggregation and centralization must be carefully distinguished, and the "conflict" above results from confusing the two.

A similar conclusion holds if we put aside the Bayesian framework and consider the more familiar problem of collecting data to (classically) estimate demands for use in a CTP. The standard result concerning aggregate demands mentioned above can be phrased as follows: Fewer observations are required to estimate total demand with a given level of precision than to estimate the individual demands with the same level of precision.

This argument fails in the present case precisely because of the need for disaggregation, which requires estimates at *some* level of precision of the individual demands. Indeed, any

disaggregation method will equate total shipments to individual destinations and their respective estimated demands, so it is the precision of the latter estimates that is important; the precision of the estimate of total demand is irrelevant. In sum, the use of an aggregate model *per se* without centralization of the system, does not reduce data requirements at all.

It is no doubt often true that a relatively simple aggregate level model is used "for planning purposes" in situations where in fact some centralization is possible. The analysis above does not apply to such cases. As is well known, to model any such centralized distribution system requires a multi-stage stochastic program. (See, e.g., [9] and the references therein.) An analysis of the effects of aggregation on such models would be most interesting, but is outside the scope of this paper.

5. BOUNDS

A Posteriori Bound: Suppose the aggregate problem (4) is defined according to fixed-weight disaggregation, and then solved. We now derive an *a posteriori* bound on $z^0 - z^*$, using the solution of (4), but not that of (1). First, the following relations are straightforward:

$$z^0 = \sum_i s_i \pi_i^{A0} + \sum_j y_j^0 \rho_{k(j)}^{A0} + \sum_j a_j(y_j^0),$$

$$z^* = \sum_{ij} c_{ij} x_{ij}^* + \sum_j a_j(y_j^*),$$

$$0 \geqslant \sum_i \pi_i^{A0}\left(s_i - \sum_j x_{ij}^*\right) + \sum_j \rho_{k(j)}^{A0}(y_j^* - \sum_i x_{ij}^*).$$

Combining these expressions, and rearranging terms, we obtain

$$\begin{aligned} z^0 - z^* \leqslant & \sum_i s_i \pi_i^{A0} - \sum_i s_i \pi_i^{A0} \\ & + \sum_{ij} (\pi_i^{A0} + \rho_{k(j)}^{A0} - c_{ij}) x_{ij}^* \\ & + \sum_j y_j^0 \rho_{k(j)}^{A0} + \sum_j a_j(y_j^0) \\ & - \sum_j y_j^* \rho_{k(j)}^{A0} - \sum_j a_j(y_j^*). \end{aligned}$$

The first two terms cancel. The next term is less than or equal to

$$\sum_j (\rho_{k(j)}^{A0} + d_j) \sum_i x_{ij}^* = \sum_j (\rho_{k(j)}^{A0} + d_j) y_j^*,$$

where $d_j = \max_i \{\pi_i^{A0} - c_{ij}\}$, $j \in J$. Thus,

$$\begin{aligned} z^0 - z^* \leqslant & \sum_j y_j^0 \rho_{k(j)}^{A0} + \sum_j a_j(y_j^0) \\ & + \sum_j y_j^* d_j - \sum_j a_j(y_j^*), \end{aligned} \tag{9}$$

so

$$\begin{aligned} z^0 - z^* \leqslant & \sum_j [y_j^0 \rho_{k(j)}^{A0} + a_j(y_j^0)] \\ & - \sum_j \min \{-d_j y_j + a_j(y_j) : y_j \geqslant 0\}. \end{aligned} \tag{10}$$

This is our bound. The solution of the jth minimization is finite, since

$$\pi_i^{A0} - c_{ij} \leqslant -c_{ij} \leqslant -\delta_j, \quad i \in I,$$

so

$$\gamma_j + d_j \leqslant \gamma_j - \delta_j.$$

The minimum is achieved at

$$\inf \{y_j : F_j(y_j) \geqslant (\gamma_j + d_j)/(\gamma_j - \delta_j),\ y_j \geqslant 0\},$$

so the bound requires relatively little extra computation.

Under the second form of optimal disaggregation, suppose that our approximation of q_k is exact at the optimum, and in particular, the w_j used to construct the aggregate objective function are exactly correct. Then, the expression above for z^0 holds, so the bound (10) applies in this case also. Even when these conditions cannot be ensured, computation of the right-hand-side of (10) may provide a useful, if approximate, measure of suboptimality.

A Priori Bound: We now derive a bound which does not require problem (4) to have been solved, assuming (A1) and (A2). For this purpose we first require *a priori* upper and lower bounds on y^* itself:

First, y_j^* is certainly no greater than would be the case if we could ship without restriction from the source having minimal shipping cost to j. Also, y_j^* is limited by total supplies. Thus $y_j^* \leqslant y_j^+$ where y_j^+ is the smaller of $\sum_i s_i$ and

$$\inf \{y_j : y_j \geqslant 0,\ F_j(y_j) >= (\gamma_j - \min_i\{c_{ij}\})/(\gamma_j - \delta_j)\}.$$

To obtain a lower bound, let $c^+ = \max_{ij}\{c_{ij}\}$ and let

$$y_j^- = \inf \{y_j : F_j(y_j) \geqslant (\gamma_j - c^+)/(\gamma_j - \delta_j),\ y_j \geqslant 0\}.$$

Then, y_j^- is a lower bound on y_j^*, *if* the sum of the y_j^- thus computed is no greater than $\sum_i s_i$. Actually, these lower bounds can be considerably improved, if we can determine *a priori* that certain $x_{ij}^* = 0$. Letting S be the set of (i, j) which *cannot* be so eliminated, redefine $c^+ = \max \{c_{ij} : (i, j) \in S\}$. Again the y_j^- derived from this c^+ are valid lower bounds, subject to the condition mentioned.

Proceeding to the bound on $z^0 - z^*$, let us rewrite (9) as follows:

$$z^0 - z^* \leqslant \sum_j \left[-(\gamma_j - \rho_{k(j)}^{A0})y_j^0 + (\gamma_j - \delta_j) \int_{-\infty}^{y_j^0} F_j(\xi)\, d\xi\right] \tag{11}$$

$$- \sum_j \left[-(\gamma_j + d_j)y_j^* + (\gamma_j - \delta_j) \int_{-\infty}^{y_j^*} F_j(\xi)\, d\xi\right].$$

The first summation equals

$$\sum_k \left[-(\gamma_k^A - \rho_k^{A0})y_k^{A0} + (\gamma_k^A - \delta_k^A) \int_{-\infty}^{y_k^{A0}} F_k^A(\xi)\, d\xi\right]. \tag{12}$$

But, under (A1) and (A2), for each k,

$$\int_{-\infty}^{y_k^{A0}} F_k^A(\xi)\,d\xi = \sum_{T_k} w_j \int_{-\infty}^{y_k^{A0}} F_k^A(\xi)\,d\xi = \sum_{T_k} \int_{-\infty}^{y_j^0} F_j(\xi)\,d\xi,$$

so (12) equals

$$\sum_j \left[-(\gamma_{k(j)}^A - \rho_{k(j)}^{A0}) y_j^0 + (\gamma_{k(j)}^A - \delta_{k(j)}^A) \int_{-\infty}^{y_j^0} F_j(\xi)\,d\xi \right]. \tag{13}$$

Moreover,

$$F_j(y_j^0) = F_{k(j)}^A(y_{k(j)}^{A0}),\ j \in J,$$

so in fact y_j^0 minimizes the jth term in (13). Thus, if we substitute y_j^* for y_j^0 in (13), the inequality still holds. Also,

$$\begin{aligned} \gamma_j + d_j &= \gamma_j - \rho_{k(j)}^{A0} + \max_i \{\rho_{k(j)}^{A0} + \pi_i^{A0} - c_{ij}\} \\ &\leqslant \gamma_j - \rho_{k(j)}^{A0} + e_j, \end{aligned}$$

where $e_j = \max_i \{c_{ik(j)}^A - c_{ij}\}$, by the optimality conditions for the aggregate problem. In sum, from (11) we obtain

$$\begin{aligned} z^0 - z^* \leqslant \sum_j \Bigg\{ & -(\gamma_{k(j)}^A - \gamma_j - e_j) y_j^* \\ & + [(\gamma_{k(j)}^A - \gamma_j) - (\delta_{k(j)}^A - \delta_j)] \int_{-\infty}^{y_j^*} F_j(\xi)\,d\xi \Bigg\}, \end{aligned}$$

so

$$\begin{aligned} z^0 - z^* \leqslant \sum_j \max \Bigg\{ & -(\gamma_{k(j)}^A - \gamma_j - e_j) y_j \\ & + [(\gamma_{k(j)}^A - \gamma_j) - (\delta_{k(j)}^A - \delta_j)] \int_{-\infty}^{y_j} F_j(\xi)\,d\xi : y_j^- \leqslant y_j \leqslant y_j^+ \Bigg\}. \end{aligned} \tag{14}$$

Expression (14) is the *a priori* bound. In the jth maximand, if the term in square brackets is negative, the maximand is concave, and the maximum is easily computed. If this term is nonnegative, the maximand is convex, and the maximum is achieved at one of the endpoints.

A qualitative analysis of (14) suggests a criterion for aggregation which is consistent with earlier work on deterministic problems [5, 13]: Suppose the costs $\{c_{ij} : j \in T_k\}$ are nearly identical for each i, and that each set $\{\gamma_j : j \in T_k\}$ and $\{\delta_j : j \in T_k\}$ has nearly identical elements as well. Then, each constant in parentheses in (14) is close to zero for $j \in T_k$, so these terms contribute little to the overall bound. This observation suggests aggregating destinations whose corresponding cost coefficients are all similar. The size of the demands at the respective destinations appears to be irrelevant.

A Posteriori Bound for Optimal Disaggregation, First Form: Williams [12] derives a bound on the suboptimality in (1) of any y, together with the optimal solution of the CTP with demands y. Clearly, this result can be applied to (x^0, y^0), derived from the solution to (5) by the first form of optimal disaggregation described above. Let z' be the total cost of this solution, and (π', ρ') the optimal dual solution of the CTP. Then, the bound is as follows:

$$z' - z^* \leqslant \sum_j \left[-(\gamma_j - \rho_j')(y_j^0 - y_j') + (\gamma_j - \delta_j) \int_{y_j'}^{y_j^0} F_j(\xi)\, d\xi \right], \tag{15}$$

where $y_j' = \inf \{y_j : y_j \geqslant 0,\ F_j(y_j) = (\gamma_j - \rho_j')/(\gamma_j - \delta_j)\}$, $j \in J$. Note the similarity in form (and hence computational effort) of (15) and the other bounds derived above. Under assumption *d*, a weaker bound, which is even simpler to compute however, is also due to Williams [12]:

$$z' - z^* \leqslant \sum_j (\gamma_j - \delta_j)(y_j^0 - y_j')[F_j(y_j^0) - F_j(y_j')]. \tag{16}$$

It would be useful to be able to compute an *a priori* bound on $z' - z^*$. Since $z' \leqslant z^0$, the bound (14) applies to $z' - z^*$. We have not found a stronger such result.

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A CLASS OF PARAMETRIC PROPERTIES OF TRANSPORTATION PROBLEMS

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ABSTRACT

This paper studies certain "second order" parametric relations in capacitated transportation problems. These relations concern the question of what happens to the effect of a parameter (first derivative) as another parameter is varied. These relationships have been found quite useful in the solution of many types of facility location and capacity expansion problems. The paper presents several results on the parametric behavior of the dual multipliers from which second order parametric relations can be derived.

1. INTRODUCTION

This paper is concerned with the study of certain "second order" parametric relations in capacitated transportation problems:

$$(P): \quad \text{Min } Z = \sum_{i,j} c_{ij} x_{ij}$$

$$(1) \quad \text{Subject to } \sum_{j} x_{ij} \leqslant S_i \quad i = 1 \ldots, n$$

$$(2) \quad \sum_{i} x_{ij} = D_j \quad j = 1 \ldots, n$$

$$(3) \quad L_{ij} \leqslant x_{ij} \leqslant M_{ij} \quad i = 1 \ldots, n \quad j = 1 \ldots, m.$$

In the above formulation c_{ij} denotes unit shipping costs; S_i and D_j are the magnitudes of availabilities and requirements, respectively; and L_{ij} and M_{ij} ($\geqslant 0$) represent bounds on the shipments, X_{ij}. Specification of the way in which Z changes with respect to the constants of (P) is commonly known as parametric analysis. To discuss different types of parametric relationships let α_k stand for any one of the constants (data) of *(P)* which is being parameterized, and $Z(\alpha_k)$ be the optimal cost function. A common way to generate this function is to evaluate $\partial Z/\partial \alpha_k$ for various α_k levels while holding all other data fixed. We characterize this well known relationship between α_k and $Z(\alpha_k)$ as a *simple parametric* relation. If we now allow another problem data to vary also, and denote its various levels by α_r, we may define a *second-order* parametric relationship (SOPR, henceforth) as the specification of how a change in α_r affects the function $Z(\alpha_k)$ over its entire domain. The "shift" in $Z(\alpha_k)$ due to a change in α_r

may be specified through the derivative of $\partial Z/\partial\alpha_k$ with respect to α_r, i.e., through $\partial^2 Z/\partial\alpha_k\,\partial\alpha_r$. The well-known result that $Z(\alpha_k)$ is convex in α_k is an example of a SOPR, in the above sense, since the convexity of $Z(\alpha_k)$ is equivalent to the statement: $\dfrac{\partial^2 Z}{\partial\alpha_k^2} \geqslant 0\ \forall\ \alpha_k$. However, this paper considers more general relationships in the form of $\partial^2 Z/\partial\alpha_k\,\partial\alpha_r$ where $k \neq r$, from which further useful results may be derived.

We were not able to locate many papers which directly address the SOPR in transportation problems. Erlenkotter [4] gives several specific SOPR results in the uncapacitated transportation problem. We will discuss some of these results in Section 3. Another relevant work is by Shapley [10] who studied similar relationships occuring in assignment problems.

The ability to judge, without re-solving the problem, whether the changes in two (or more) parameters *reinforce, dampen* or *leave unchanged* one another's individual effects may be quite useful in post-optimality analyses of (P). As we shall demonstrate later, such interactive effects can be derived from the SOPR in a straightforward way. An even more important use of these interactive effects can be made in the study of large problems in which some version of (P) appears as a sub-problem. For example, in determining and optimal capacity expansion schedule for a multi-plant system [4,5,6], it is imperative to know how a change in the capacity of one source affects the potential benefits of expanding the capacities of some others. A similar situation exists in solving the capacitated facility location problems [1,8,9]. One needs to take into account the fact that potential benefits of adding new facilities or dropping existing ones depend on the current set of facilities.

In the next section we introduce some definitions and preliminaries on the structure of (P). Section 3 gives several specific SOPR and shows how these may be used to determine the interactive effects of simultaneous parameter changes.

2. PRELIMINARIES

Since $\partial Z(\alpha_k)/\partial\alpha_k$ can be interpreted as a dual multiplier, a SOPR can be studied from the parametric behavior of the dual multipliers. For instance, if S_k and S_r are the two problem data in (P) for which a SOPR is to be studied, and if u_k denotes the dual multiplier associated with the k^{th} supply constraint, the SOPR of interest is simply the function $\dfrac{\partial}{\partial S_r}\left[\dfrac{\partial Z}{\partial S_k}\right]$ or equivalently $\partial u_k/\partial S_r$.

In order to facilitate the analysis of the behavior of the dual multipliers, some well known graph theoric properties of the solutions to (P) and some definitions are given below:

Let B be a set of arcs (ij) constituting a feasible basis for (P). B and the corresponding flows x_{ij} are optimal to (P) if nonnegative dual multipliers u_i and v_j can be found respectively for Equations (1) and (2) of (P) satisfying [3, p. 377]:

$$u_i > 0 \rightarrow \sum_j x_{ij} = S_i \tag{4}$$

$$v_j - u_i > c_{ij} \rightarrow x_{ij} = M_{ij} \tag{5}$$

$$v_j - u_i < c_{ij} \rightarrow x_{ij} = L_{ij} \tag{6}$$

$$v_j - u_i = c_{ij} \rightarrow L_{ij} \leqslant x_{ij} \leqslant M_{ij}. \tag{7}$$

Arcs satisfying (5) and (6) are *nonbasic* at their *upper* and *lower* bounds respectively while those satisfying (7) are *basic*. Given the dual multipliers u_i and v_j, the dual multipliers t_{ij} and l_{ij} associated with restrictions $x_{ij} \leqslant M_{ij}$ and $x_{ij} \geqslant L_{ij}$ are implicit in (4)-(7) and can be obtained as:

$$t_{ij} = \text{Max } \{-\sigma_{ij},\ 0\}$$

$$l_{ij} = \text{Max } \{\sigma_{ij},\ 0\}$$

where $\sigma_{ij} = c_{ij} + u_i - v_j$.

If B is a basis, then its columns form a spanning forest consisting of spanning trees each corresponding to nonbinding Equation (2) [7]. However if a set of slack variables is associated with a dummy, $(m + 1)^{st}$ sink, then each basis of (P) corresponds to a unique spanning tree which connects every node (supply and demand points) to every other node without any cycles. Every nonbasic arc can be represented in terms of the tree arcs by using a $(+1, -1, 0)$ valued vector. An exchange of a basic and a nonbasic arc yields an adjacent basis, i.e., for $(k,r) \in B^0$ and $(i_b, j_b) \notin B^0$, $B^1 = B^0 - \{(k,r)\} \cup \{(i_b, j_b)\}$ is an adjacent basis if it is a basis. With respect to two adjacent bases B^0 and B^1, define two arc sets:

$$B^+ = B^0 \cup B^1$$

$$B^- = B^0 \cap B^1.$$

The union of the two bases, B^+ has a unique cycle of arcs denoted by C whose member arcs are numbered sequentially beginning with the leaving arc (r,k). The intersection of the two bases, B^- divides the nodes into two groups (as shown by the vertical bar in Figure 1): Those nodes which are connected to node r by some arcs in B^-; and those which are connected to node k by some arcs in B^-. These node sets are denoted by N^r and N^k respectively.

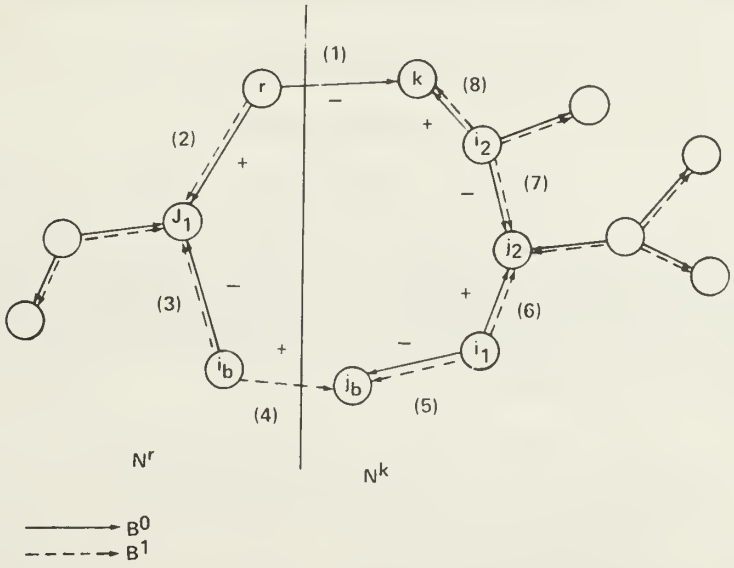

FIGURE 1a. — The incoming arc is even numbered

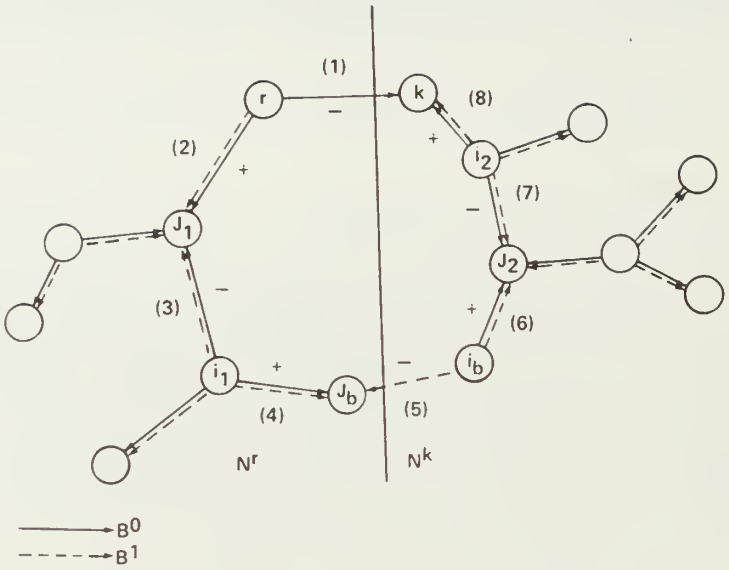

FIGURE 1b. — The incoming arc is odd numbered

3. PARAMETRIC RELATIONSHIPS

The first part of this section will examine the parametric behavior of the dual multipliers with respect to a problem constant. We will, then, show how these results may be used to derive the interactive effects when certain problem data are parameterized simultaneously.

THEOREM 1: When an arc capacity, say M_{rk}, is reduced, then:

(i) t_{rj}, for all j, and t_{ik} for all i cannot decrease;

(ii) l_{rj}, for all j, and l_{ik} for all i cannot increase.

PROOF: Since any magnitude of change can be considered as a sum of smaller changes, it is sufficient to prove the result for some Δ (a reduction in M_{rk}) which is of sufficient magnitude to require exactly *one* basis change for restoring optimality. Reducing M_{rk} by Δ renders the current optimal solution x_{ij}^0 infeasible. To restore optimality the dual simplex algorithm removes arc, (r,k) and locates a new arc, (i_b, j_b) to enter the basis. The choice of Δ and the entry criterion of the dual simplex algorithm guarantee optimal primal and dual solutions. The primal solution, x_{ij}^1 corresponding to the new basis B^1 is:

$$x_{ij}^1 = \begin{cases} x_{ij}^0 + \Delta \text{ for } (i,j) \in C, \text{ and even numbered} \\ x_{ij}^0 - \Delta \text{ for } (i,j) \in C, \text{ and odd numbered} \\ x_{ij}^0 \text{ for } (i,j) \notin C \end{cases}$$

as can be seen from Figure 1a or 1b.

CASE (a): If the incoming arc is an odd numbered member of C, then $x^0_{i_b j_b} = M_{i_b j_b}$ and $\sigma_{i_b j_b} < 0$ (see Fig. 1a). To satisfy (7) for (i_b, j_b), $\sigma_{i_b j_b} = c_{i_b j_j} + u_{i_b} - v_{j_b}$ must be increased to 0. If $(m+1) \in N^r$, then u_{i_b} must increase, otherwise v_{j_b} should decrease to increase $\sigma_{i_b j_b}$ and preserve (4). We may assume u_{i_b} increases by $\delta = -\sigma_{i_b j_b}$ without loss of generality for if (4) is violated by this, (δ) can be subtracted from all u_i and v_j to restore (4) leaving all t_{ij} and l_{ij} intact. Furthermore, to maintain (7) on the other basic arcs, the multipliers of all the nodes in N^k must increase by δ since $i_b \in N^k$.

CASE (b): If (i_b, j_b) is an even numbered arc of C then $x^0_{i_b j_b} = L_{i_b j_b}$ and $\sigma^0_{i_b j_b} > 0$ (see Fig. 1b). To satisfy (7) for (i_b, j_b), v_{j_b} must increase by $\delta = \sigma_{i_b j_b}$ (again assume (4) remains valid with no loss of generality). Also to maintain (7) for the other basic arcs, the multipliers of all the nodes in N^k must increase since $j_b \in N^k$.

$$u_i^1 = \begin{cases} u_i^0 + \delta & i \in N^k, \\ u_i^0 & i \in N^r; \end{cases} \quad \text{and} \quad v_j^1 = \begin{cases} v_j^0 + \delta & j \in N^k, \\ v_j^0 & j \in N^r. \end{cases}$$

Consequently, the definitions of t_{ij} and l_{ij} lead to:

$$t^1_{rj} = \begin{cases} \text{Max } \{-\sigma^0_{rj} + \delta, 0\} \geqslant t^0_{rj} & j \in N^k \\ t^0_{ij} & j \in N^r \end{cases}$$

$$t^1_{ik} = \begin{cases} t^0_{ik} & i \in N^k \\ \text{Max } \{-\sigma^0_{ik} + \delta, 0\} \geqslant t^0_{ik} & i \in N^r \end{cases}$$

$$l^1_{rj} = \begin{cases} \text{Max } \{\sigma^0_{rj} - \delta, 0\} \leqslant l^0_{rj} & j \in N^k \\ l^0_{rj} & j \in N^r \end{cases}$$

$$l^1_{ik} = \begin{cases} l^1_{ik} & i \in N^k \\ \text{Max } \{\sigma^0_{ik} - \delta, 0\} \leqslant l^0_{ik} & i \in N^r \end{cases}$$

THEOREM 2: If a lower bound, L_{rk} is reduced, then

(i) t_{rj}, for all j, and t_{ik} for all i cannot increase;

(ii) l_{rj} for all j, and l_{ik} for all i cannot decrease.

PROOF: Using the same argument as before, we conclude that when the flow is forced up on arc (r, k) it leaves the basis by reaching its upper bound and either an even numbered arc increases its flow beyond its lower bound or an odd numbered arc reduces its flow below its upper bound to become basic. In either case, u_i, $i \in N^r$ and v_j, $j \in N^r$ increase by δ. Definitions of t_{ij} and l_{ij} lead to:

$$t^1_{rj} = \begin{cases} \text{Max } \{-\sigma^0_{rk} - \delta, 0\} \leqslant t^0_{rj} & j \in N^k \\ t^0_{rj} & j \in N^r \end{cases}$$

$$t^1_{ik} = \begin{cases} t^0_{ik} & i \in N^k \\ \text{Max } \{-\sigma^0_{ik} - \delta, 0\} \leqslant t^0_{ik} & i \in N^r \end{cases}$$

$$l_{rj}^1 = \begin{cases} \text{Max } \{\sigma_{rj} + \delta,\ 0\} \geqslant l_{rj}^0 & j \in N^k \\ l_{rj}^0 & j \in N^r \end{cases}$$

$$l_{ik}^1 = \begin{cases} l_{ik}^0 & j \in N^k \\ \text{Max } \{\sigma_{ik} + \delta,\ 0\} \geqslant l_{ik}^0 & j \in N^r \end{cases}$$

The above results establish that t_{ij} and l_{ij} are monotonic in M_{rk} and L_{rk}, if either $i = r$ or $j = k$. Multipliers t_{ij}, l_{ij}, u_i and v_j may not be monotonic for some $i \neq r$ and some $j \neq k$ since at successive iterations of the dual simplex algorithm, i and j may not consistently belong to the same sets N^r or N^k and hence u_i or v_j may increase, causing t_{ij} and l_{ij} to vacillate, at different iterations.

We now show that a result due to Erlenkotter [4] can be viewed as special cases of Theorems 1 and 2. This result concerns the behavior of dual multipliers for the uncapacitated problem when the supply and demand figures are altered.

THEOREM 3: (Erlenkotter [4]) The optimal dual multipliers u_i and v_j for a transportation problem ((P) without Equation (3)) are nondecreasing, if either.

(a) D_j are constant, and S_i decrease, or

(b) S_i are constant, and D_j increase.

PROOF: To prove (a) let $j = m + 1$ represent the slack column of the transportation problem. The definition of l_{ij} leads to $l_{i,m+1} = u_i$, reducing S_k can be viewed as increasing $L_{k,m+1}$. From Theorem 2 the result follows.

To prove (b) let $i = n + 1$ be an artificial source with $c_{n+1,j} = 0$, $M_{n+1,j} = 0\ \forall j$, and $S_{n+1} = \infty$. This modified problem is equivalent to the original one. The definition of t_{ij} leads to $t_{n+1,j} = v_j$. Reducing D_k can be viewed as increasing $M_{n+1,k}$. From Theorem 1 the result follows.

This proof also extends Erlenkotter's [4] results to the capacitated transportation problems.

Using the same reasoning employed in Theorems 1 and 2 one can study the behavior of t_{ij} and l_{ij} with respect to changes in c_{rk}. Decreases in c_{rk} can be viewed as increases in M_{rk} since in either case the condition which triggers adjustments in the primal and dual solutions is $v_k - u_r > c_{rk}$ and $x_{rk} < M_{rk}$ (violation of (5)). Conversely, increases in c_{rk} can be considered as reductions in L_{rk} since in either case $v_k - u_r < c_{rk}$ and $x_{rk} > L_{rk}$ (violation of (6)).

Having established the behavior of the duals, second order parametric properties can be derived. We will give such a derivation using a procedure first employed by Erlenkotter [4] to prove that $Z(S_i) - Z(S_i + a)$ is nonincreasing in another capacity, S_j for $a \geqslant 0$. Theorem 4 establishes, as an example of derivation of interactive effects, the behavior of the cost difference in (P) due to changing the capacity of one arc, as a function of a second arc capacity. Notation $Z(a,b)$ and t_{ij} (a,b) denotes the optimal cost of (P) and the value of t_{ij} when the capacities of the two arcs are a and b respectively.

THEOREM 4: The increase in $Z(.\ ,\ .)$ due to tightening $x_{rk} \leqslant M_{rk}$ is nondecreasing as (i) $x_{ik} \leqslant M_{ik}$ for any i or (ii) $x_{rj} \leqslant M_{rj}$ for any j is tightened; or equivalently for any given α and $\beta \geqslant 0$,

$$\text{(i)}\quad Z(M_{rk}-\alpha, M_{rj}) - Z(M_{rk}, M_{rj}) \leqslant Z(M_{rk}-\alpha, M_{rj}-\beta) - Z(M_{rk}, M_{rj}-\beta)$$

$$\text{(ii)}\quad Z(M_{rk}-\alpha, M_{ik}) - Z(M_{rk}, M_{ik}) \leqslant Z(M_{rk}-\alpha, M_{ik}-\beta) - Z(M_{rk}, M_{ik}-\beta).$$

PROOF: We will only prove (i) since the proof for (ii) is almost identical. From the well known properties of the duals we have.

$$\frac{\partial Z}{\partial M_{ij}} = -t_{ij}.$$

Therefore,

$$Z(M_{rk}-\alpha,\ M_{rj}) = Z(M_{rk},\ M_{rj}) + \int_0^{\alpha} t_{rk}(M_{rk}-\delta,\ M_{rj})\,d\delta$$

$$Z(M_{rk},\ M_{rj}-\beta) = Z(M_{rk},\ M_{rj}) + \int_0^{\beta} t_{rj}(M_{rk},\ M_{rj}-\delta)\,d\delta$$

and

$$Z(M_{rk}-\alpha,\ M_{rj}-\beta) = Z(M_{rk},\ M_{rj}) + \int_0^{\alpha} t_{rk}\,(M_{rk}-\delta,\ M_{rj})\,d\delta$$
$$+ \int_0^{\beta} t_{rj}(M_{rk}-\alpha,\ M_{rj}-\delta)\,d\delta.$$

Substituting these expressions in the statement of (i) we get,

$$\int_0^{\beta} [t_{rj}(M_{rk}-\alpha,\ M_{rj}-\delta) - t_{rj}(M_{rk},\ M_{rj}-\delta)]\,d\delta \geqslant 0.$$

This is true, since by Theorem 1 we have

$$t_{rj}(M_{rk},\ M_{rj}-\delta) \leqslant t_{rj}\,(M_{rk}-\alpha,\ M_{rj}-\delta) \text{ for } 0 \leqslant \delta \leqslant \beta.$$

A corollary of the above has been applied to devise pruning tests in a branch and bound algorithm for a multi-activity facility location problem [2].

COROLLARY: The cost increase due to deleting a subset A (possibly all) of the arcs emanating from a source r is no smaller than the sum of the cost increases associated with disjoint subsets, A_l of A. i.e.,

$$Z[P|M_{rj} = 0,\ j \in A] - Z[P] \geqslant \sum_l \{Z[P|M_{rj} = 0,\ j \in A_l] - Z[P]\}$$

where notation $Z[P|\Lambda]$ denotes the optimal cost of problem (P) when condition Λ is true.

PROOF: Reducing $M_{rj},\ j \epsilon A_{l_1}$ renders the penalties due to reducing $M_{rj},\ j \epsilon A_{l_2}$ higher from Theorem 4. The result directly follows.

An important conclusion of this paper is that one can analyze the magnitude of cost difference for any two values of some parameter as a function of another by studying the parametric behavior of the dual multipliers. Consequently, general interactive effects of parameters can be determined to the extent that the parametric behavior of the relevant multipliers are monotonic.

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PRODUCTION-LOCATION PROBLEMS*

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ABSTRACT

The joint problems of determining the optimal plant location *and* optimal input mix and plant size are addressed. The interrelationship between input substitutability and plant location is stressed. Conditions under which the location problem can be separated from the determination of the optimal input mix are developed for a number of problem variations. The stability of the optimal location in the face of changes in problem parameters is also discussed. It is demonstrated that consideration of input substitutability often makes the resulting problem no more difficult to solve than problem formulations in which the inherent input substitutability is ignored.

INTRODUCTION

The joint problems facing a profit maximizing firm of determining its optimal plant location and the optimal input mix and plant size have historically been treated separately. There have been a few significant exceptions where the two problems have been treated simultaneously, at least for special cases. Perhaps the seminal effort is that of L. Moses [14], followed several years later by the efforts of A. P. Hurter and R. E. Wendell [7], A. Goldman [6], and most recently be A. P. Hurter, J. S. Martinich, and E. R. Venta [8]. Other contributions to an overall understanding of production-location problems include (but are not limited to) the works of M. Bradfield [1], R. S. Woodward [22], D. L. Emerson [4], A. Khalili, V. K. Mathur and D. Bodenhorn [9], V. K. Mathur [12] and S. M. Miller and O. W. Jensen [13]. Here, these results are unified and extended to the case in which initially a single plant is to be located on the plane and different transportation structures are employed. Later, a multiplant model is investigated. The usual assumption of a known quantity of product demanded is retained for each commodity. Demand considerations within a production-location problem context are dealt with elsewhere by E. R. Venta and A. P. Hurter [19].

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A SINGLE PRODUCT SINGLE PLANT PROBLEM — GENERAL FORMULATION

The firm is assumed to seek the plant location and the mix of inputs to utilize in order to minimize the costs of supplying known demands at each of several markets. The inputs are sold at fixed prices at their sources and are transported to the plant, thereby incurring transport costs. The output of the plant is shipped to markets of known locations, and in some instances the firm incurs transport costs for the product. The firm is constrained by its available production technology, which is summarized in the form of a production function. It is assumed that the production function can be represented by a homothetic function.*

The general problem under consideration is represented formally in P1 below:

$$\text{(1.0)} \qquad \min \sum_m C_p(x,\ b_m,\ q_m) + \sum_j C(x,\ a_j,\ z_j)$$

(P1)

$$\text{(1.1)} \qquad \text{subject to } \sum_m q_m = f(z)$$

where:

$x \triangleq$ the vector variable denoting the location of the plant in R^2

$z_j \triangleq$ the quantity of the jth input used, $j = 1, \ldots, J$ — a variable

$q_m \triangleq$ the known quantity of the product demanded at market center m, $m = 1, \ldots, M$

$b_m \triangleq$ the known vector location of the mth market center in R^2

$C_p(x,\ b_m,\ q_m) \triangleq$ the cost of transporting q_m units of the product from location x to location b_m

$a_j \triangleq$ the known vector location in R^2 of the source of supply for the jth input $j = 1, \ldots, J$

$C(x,\ a_j,\ z_j) \triangleq$ the cost purchasing z_j units of input j at location a_j *and* of transporting these units from location a_j to location x

$f(\cdot) \triangleq$ the known homothetic production function

In the following section, several versions of P1 will be discussed. Each version will introduce specific assumptions about the form of the objective function (1.0) and/or the production function (1.1).

A SINGLE PRODUCT PROBLEM WITH SPECIAL TRANSPORT COSTS

Consider a simplified version of P1 in which the cost of transporting the product to market is negligible. In this case, the $C_p(x,\ b_m,\ q_m)$ terms in (1.0) will be zero. Assume that the costs of transporting inputs are constant on a per unit of input per unit of distance basis. Then problem P1 with these assumptions may be written as problem P2 below:

$$\text{(2.0)} \qquad \min \sum_j (r_j\ d(x,\ a_j) + p_j)\ z_j$$

*Notice that this formulation allows for the possibility of more than one supplier for any input. For example, in a two-input Cobb-Douglas technology

$$f(z) = A(z_{11} + z_{12})^{\delta}(z_2)^{(1-\delta)}$$

with z_{11} and z_{12} as quantities of input one supplied from alternate sources.

(P2)

(2.1) subject to $z_0 = f(z)$

where:

$p_j \triangleq$ the price per unit of input j at its source a_j

$d(x, a_j) \triangleq$ the "distance" between locations a_j and x

$r_j \triangleq$ the cost of transporting a unit of input j a unit of distance

$z_0 \triangleq \sum_m q_m$

A. P. Hurter and R. E. Wendell [7] introduced Lagrangian optimization to the analysis of this problem and demonstrated that, with a certain class of production functions, one could first solve for the optimal plant location, x, and then for the optimal input mix, z, without fear of sub-optimization. A. Goldman [6] later identified the class of production functions used by Hurter and Wendell as the class of all homothetic production functions. Goldman also demonstrated that solution algorithms that have been developed for the Weber location problem (i.e., P2 with the z_j all fixed) can be used to solve for the optimal location in P2. Hurter, Martinich and Venta [8] have demonstrated that the optimal location is invariant with parametric changes in z_0.

A more convenient form of the problem, which allows a more direct exploitation of the properties of homothetic production function, is now developed and used throughout the remainder of the paper.

Define:

$$C(x, z_0) = \min \sum_j (r_j d(x, a_j) + p_j) z_j$$

subject to $z_0 = f(z)$

$C(x, z_0)$ is then the minimum cost of producing z_0 units of the product at location x. $C(x, z_0)$ will be referred to as the "cost function."

LEMMA 1: Let f be a homothetic production function.

Then:

(*) $$C(x, z_0) = h(x)\, g(z_0)$$

PROOF: By definition, a homothetic function $f(z)$ can be written as $f(z) = F(H(z))$, where H is homogeneous of degree one and F is an increasing, differentiable transformation.

Let $h(x) = \min \sum_j (r_j d(x, a_j) + p_j) z_j$

subject to $1 = H(z)$

Since H is homogeneous of degree 1, the minimum cost of producing z_0 units at x using technology H is $C_H(x, z_0) = h(x) z_0$. But

$$H(z) = F^{-1}(z_0) \triangleq g(z_0), \text{ thus } C(x, z_0) = h(x)\, g(z_0). \qquad \square$$

Lemma 1 is a spatial version of Shephard's [7] duality theorem for homothetic production functions.

THEOREM 1: If the production function in P2 is homothetic, then a single optimal location exists for all output rates.*

PROOF: Using Lemma 1 (P2) can be rewritten as min $h(x)\ g(z_0)$ which implies that solving min $h(x)$ will yield the optimal location. □

As the result of Theorem 1, P2 can be solved directly. As a first step, the form of the marginal cost function, $h(x)$, must be determined. this function is minimized with respect to the vector location variable, x. The solution, x^*, is optimal for all levels of output. Solving P2 with $x = x^*$ will yield the optimal mix.†

Embedded within this solution procedure is the solution of a location problem min $h(x)$. As in the ordinary or Weber location problem, the solution scheme must be iterative in nature. The basic solution scheme for the Weber problem was developed by L. Cooper [3] and, independently, by H. W. Kuhn and R. E. Kuenne [10] for convex objective functions. A. Goldman [6], in addition to pointing out that $h(x)$ generally is not convex, extended the scheme developed by Kuhn and Kuenne [10] to account for local minima at singular points (i.e., the sources of the inputs), and provided a converging fixed point iterative algorithm to find local minima at other than singular points. It appears then that solving min $h(x)$ is not significantly more difficult then solving the Weber problem.

Next consider a case in which the firm must pay for the transport of its product to market, so that the $C_p(x,\ b_m,\ q_m)$ terms of P1 are no longer zero as they were in P2. Instead, let $C_p(x,\ b_m,\ q_m) = rd(x,\ b_m)q_m$. Then, P1 can be written:

$$\min \sum_m rd(x,\ b_m)\ q_m + \sum_j (r_j d(x,\ a_j) + p_j))z_j \tag{3.0}$$

(P3)

$$\text{subject to } \sum_m q_m = z_0 = f(z) \tag{3.1}$$

Making use of Lemma 1, P3 can be rewritten as

$$\min_x \sum_m r\ d(x,\ b_m)\ q_m + h(x)\ g(z_0) \tag{3.2}$$

The objective (3.2) contains only the location variable x and thus, as in the case of problem P2, it establishes a separability between production and location decisions so that first a problem involving only the location variable can be solved and subsequently the optimal input mix can be determined. However, in problem P3 the objective (3.2) includes both the level of output, z_0, and the demands at each market, q_m. In general, the optimal location will change as the q_m and/or z_0 change.

*Theorem 1 was proved in [8] in a different form and is included here for completeness.

†The explicit determination of $h(x)$ is necessary for the solution procedure described. $h(x)$ has been derived explicitly for Cobb-Douglas functions in [7] and for CES functions in [6]. Goldman [6] also gives general conditions for determining $h(x)$. However, even if $h(x)$ cannot be determined explicitly, the characterization of the cost function given in Lemma 1 is useful in developing properties of the location decisions.

If the production function, f, is homogeneous of degree one, then $g(z_0)$ and z_0 are the same in the preceding formulation and (3.2) can be written as:

$$\min \sum_m (r\ d(x,\ b_m) + h(x))\ q_m. \tag{3.3}$$

The optimal location is now insensitive to the level of output as long as demands increase or decrease proportionally over all markets (i.e., as long as q_i/q_j is constant for all $i, j = 1, \ldots, m$). Note that in the case of only one market the same location is optimal for all output levels. To summarize:

THEOREM 2: If the production function in P3 is homothetic, then the production-location problem can be written as a problem in the location variable only given by (3.2). Further, if the production function is homogeneous of degree one and any change in output level is distributed proportionally to all markets, then a single location will be optimal for all output levels.

EXTENSION OF THE SINGLE PLANT PROBLEM

Consider problem P2 with nonlinear (in distance) transportation costs, so that terms of the form $r_j\ d(x,\ a_j)$ are replaced by $T_j(d\ (x,\ a_j))$ where T_j is a general transport rate whose value depends upon the distance traveled. Then the problem takes the form:

$$\min \sum_j [T_j(d(x,\ a_j)) + p_j]z_j \tag{4.0}$$

(P4)

$$\text{subject to } z_0 = f(z) \tag{4.1}$$

The form of this problem is similar to that encountered earlier and Lemma 1 applies, yielding a cost function in the form of (*). Consequently, Theorem 1 applies. As the form of $T_j[d(x,\ a_j)]$ increases in complexity, the difficulty encountered in solving the location problem analogous to min $h(x)$ also increases. Conceptually, at least, the same general computational scheme can be utilized. When linear costs for the distribution of product are considered, along with nonlinear input transport costs, Theorem 2 also applies.

Under some circumstances, the prices paid for inputs at their sources may depend upon the quantity of input purchased, and then problem P2 can be written:

$$\min \sum_j (p_j(z_j) + r_j\ d(x,\ a_j))\ z_j \tag{5.0}$$

(P5)

$$\text{subject to } z_0 = f(z). \tag{5.1}$$

For a fixed but arbitrary location, $\bar{x}$, consider the minimization over the vector of input variables, z. The first order conditions are:

$$(d\ p_j/d\ z_j)\ z_j^* + p_j(z_j^*) + r_j\ d(\bar{x},\ a_j) = \mu(\bar{x})\ (\partial f/\partial z_j), \tag{5.2}$$

where the asterisk denotes the optimal value and μ is the multiplier. Multiplying (5.2) by z_j^*, rearranging and summing over j yields:

$$\sum_j (p_j\ (z_j^*) + r_j\ d(\bar{x},\ a_j))\ z_j^* = \mu(\bar{x}) \sum_j (\partial f/\partial z_j)\ z_j^* - \sum_j (d\ p_j/d\ z_j)\ z_j^{*2}. \tag{5.3}$$

Notice that the left-hand side of (5.3) has the same form as (5.0) and therefore the right-hand side of (5.3) is an equivalent form of the objective function for P5. This suggests the following theorem.

THEOREM 3: With respect to problem P5, suppose the following assumptions hold:

(a) f is homothetic

(b) the input price functions $p_j(z_j)$ all have the property that $(d\ p_j/d\ z_j)\ z_j^2 =$ constant $= C_j$ for all values of z_j in the domain of $p_j(z_j)$.

Then:

(1) The production-location problem can be written as a problem in the location variable only.

(2) A single optimal location exists for all levels of output,

(3) When linear costs of transporting output are included, as in P3, the separability of production and location problems holds, but the optimal location changes with the level of output, and

(4) When linear costs of transporting output are included, as in P3, there is a single optimum location if the production function is homogeneous of degree one and the change in output occurs proportionally over all markets.

PROOF: A. J. Goldman [6] has shown that, if f is homothetic, then $\sum_j (\partial f/\partial z_j)\ z_j = K(z_0)$ for all z such that $z_0 = f(z)$. This and the second assumption allow reduction of the right-hand side of (5.3) to:

$$\min_x \mu(x)\ K(z_0) + C \approx \min_x \mu(x), \tag{5.4}$$

where $\sum_j C_j \triangleq C$. Thus, results (1) and (2) are proven. Results (3) and (4) also follow directly from the arguments used in developing Theorem 2. □

A question of interest is the applicability of the second assumption in the statement of Theorem 3. The differential equation

$$(d\ p_j/d\ z_j)\ z_j^2 = C_j \tag{5.5}$$

has the solution:

$$p_j(z_j) = c_{1j}/z_j + c_{2j} \text{ or } p_j(z_j)\ z_j = c_{1j} + c_{2j}\ z_j. \tag{5.6}$$

This suggests that the cost of purchasing some positive quantity of input ($z_j > 0$) can be divided into a fixed cost c_{1j} associated with any purchase independent of size and a variable cost c_{2j} for every unit purchased. This is not an usual scheme for setting a price. In addition, if the input demand function can be scaled so that $c_{2j} = 0$, then $p_j(z_j) = c_{1j}/z_j$ or $p_j(z_j)\ z_j = c_{1j}$, a constant elasticity demand function. Over all, (5.6) can be interpreted as a pricing mechanism that offers quantity discounts in terms of a fixed cost and a constant unit cost.

Now consider P2 with transportation costs that depend on the quantity shipped. Then P2 takes the form:

$$\min \sum_j (p_j z_j + T_j(z_j, x)) \tag{6.0}$$

(P6)

$$\text{subject to } z_0 = f(z) \tag{6.1}$$

where $T_j(z_j, x)$ is the cost of transporting z_j units from the source at a_j to the plant at x. For arbitrary fixed $\bar{x}$, the first order conditions are:

$$p_j + \partial T_j/\partial z_j = \mu(\bar{x})\ (\partial f/\partial z_j) \quad j = 1, \ldots, J \tag{6.2}$$

Multiplying (6.2) by z_j^*, summing over all j, rearranging and adding $\sum_j T_j(z_j^*, \bar{x})$ to both sides, yields:

$$\sum_j (p_j z_j^* + T_j(z_j^*, \bar{x})) = \mu(\bar{x}) \sum_j (\partial f/\partial z_j) z_j^* - \sum_j (\partial T_j/\partial z_j) z_j^* + \sum_j T_j(z^*, x). \tag{6.3}$$

The left-hand side of (6.3) is similar to (6.0); so the right-hand side of (6.3) is an equivalent form of the objective function for problem P6. Since f is homothetic, the right-hand side of (6.3) can be made independent of z^* if it is further assumed that:

$$-(\partial T_j/\partial z_j)\ z_j^* + T_j(z^*, \bar{x}) = k_j \text{ for all}_j = 1, \ldots, J \text{ where} \tag{6.4}$$

the k_j are constant. Under these conditions, statements (1), (2), (3), and (4) of Theorem 3 apply and the proofs are identical.

The assumption (6.4) is analogous to that in statement (b) of Theorem 3. The equation (6.4) has a solution of the form $T = t_1 z + t_0$ where t_1 and t_0 are constants, and an interpretation of $T_j(z_j, x)$ in terms of a fixed cost plus a constant unit cost multiplied by a quantity transported is suggested. It is not unusual to find that transportation costs per unit distance can be approximated by a fixed cost per shipment regardless of quantity shipped plus a cost depending linearly on the quantity to be shipped.

A MULTIPLANT, SINGLE PRODUCT PROBLEM - GENERAL FORMULATION

When the firm operates more than one plant, the complexity of the problem is substantially increased. Now, the firm must determine the location of each plant, the total output produced at each plant, the allocation of that output to markets, and the input mix to be used at each plant. The multiplant version of P1 can be written:

$$\min \sum_m \sum_i C_p(x_i, b_m, q_{im}) + \sum_i \sum_j C(x_i, a_j, z_j^i) \tag{7.0}$$

(P7)

$$\text{subject to } \sum_m q_{im} = f_i(z^i),\ i = 1, \ldots, I \tag{7.1}$$

$$\sum_i q_{im} = q_m,\ m = 1, \ldots, M \tag{7.2}$$

where:

$x_i \triangleq$ vector location of the i th plant, $i = 1, \ldots, I$

$z_j^i \triangleq$ quantity of the jth input used at plant i, $j = 1, \ldots, J$

$q_{im} \triangleq$ the amount of the single output shipped from plant i to the mth demand center, $i = 1, \ldots, I$, $m = 1, \ldots, M$.

Although I, the number of plants, has been treated as fixed in the formulation of P7, it should be interpreted as the maximum number of plants to be used, since the optimal strategy can result in some plant(s) operating at a zero level of output.

Problem P7 embodies a very difficult problem which might be called a production-location-allocation problem. The location-allocation problem has received considerable attention in the literature with varying degrees of success.* No attempt is made in this paper to propose new solution methods for the general location-allocation problem, although an algorithm is developed for a special version (P8 below). Here, the interactions between the production, location, and allocation decisions are explored.

THE SINGLE MARKET CASE

Consider a multiplant firm which serves only one market. Assume first that the firm does not incur the costs of the distribution of output. Then the C_p terms of (7.0) are zero. In addition, if the input transport costs are linear in distance and quantity, then P7 becomes:

$$\min \sum_i \sum_j (r_j\, d(x_i, a_j) + p_j)\, z_j^i \tag{8.0}$$

(P8)

$$\text{subject to } z_0^i = f_i(z_i) \quad i = 1, \ldots, I \tag{8.1}$$

P8, if the output levels at each plant z_0^i are known, reduces to the consideration of several independent single plant problems like P2, since there is no interaction between the plants. A more interesting case arises when the total quantity of product required is known but the levels of production at each plant are to be determined optimally. Thus, consider P8 with the additional restriction

$$D = \sum_i z_0^i \tag{8.2}$$

with D known but z_0^i variable.

Problem (P8) with (8.2) is a version of what might be called a production-location-allocation problem. (P7 is a more complex version). Location-allocation problems are frequently found in the literature where, generally, iterative algorithms have been developed for their solution. In this case, Lemma 1 provides a form for the cost function which does not involve the z_j^i variables, and so P8 with (8.2) is reduced to a location-allocation problem. Since a single set of optimal locations exists for all output rates (i.e., Theorem 1 applies, following Lemma 1), the location-allocation (of D to the z_0^i) problem can be separated into two problems, as illustrated below in terms of a conceptual one pass solution algorithm:

1. Compute optimal locations, x_i^*, by solving $\min_{x_i} h_i(x_i)$ for each i.

*For works on location-allocation problems consult the excellent bibliography by R. Francis and J. Goldstein [5].

2. Since location decisions are not affected by changes in output level, the solution of:

$$\min_{z_0^i} \sum_i h_i(x_i^*)\, g_i(z_0^i)$$

$$\text{subject to } \sum_i z_0^i = D$$

provides the optimal allocation of D to the z_0^{i*}

3. To obtain the optimal input mix for each plant, solve:

$$\min_{z_i} \sum_j (r_j\, d(x_i^*,\, a_j) + p_j)\, z_j^i$$

$$\text{subject to } z_0^{i*} = f_i(z^i).$$

This algorithm has been called a conceptual algorithm because, as in the single plant case, explicitly obtaining the separability of the production and location problems is not always possible.

In the important special case of a production function homogeneous of degree one, a characterization of the optimal solution can be obtained without explicitly finding $h(x)$. After optimal locations have been found in Step 1, the allocation problem can be solved on a per unit basis and the problem of Step 2 reduces to:

$$\min_{z_0^i} \sum_i C_i\, z_0^i$$

$$\text{subject to } \sum_i z_0^i = D;\ z_0^i \geqslant 0;\ i = 1, \ldots, I$$

where C_i is the cost of production and transportation per unit of output at plant i located at x_i^* determined in Step 1. Hence, the problem in step 2 becomes a linear program with one constraint, suggesting that the optimal solution will contain one z_0^i value equal to D (for some i) and the remaining z_0^k values equal to zero ($k \neq i$). Consequently, the optimal solution to P8 with (8.2) when each $f_i(z^i)$ is homogeneous of degree one, can be expected to have a single plant operating at a positive level.

Returning to P8, suppose the firm incurs costs for distribution of product which are linear in distance and quantity shipped. Then P7 becomes

(9.0) $$\min_{x_i,\, z_j^i,\, q_i} \sum_i r\, q_i\, d(x_i,\, b) + \sum_i \sum_j (r_j\, d(x_i,\, a_j) + p_j)\, z_j^i$$

(P9)

(9.1) $$\text{subject to } q_i = f_i(z^i) \quad i = 1, \ldots, I$$

(9.2) $$\sum_i q_i = D$$

Using Lemma 1, P9 is written:

(10.0) $$\min_{x_i,\, q_i} \sum_i r\, q_i\, d(x_i,\, b) + \sum_i h_i\,(x_i)\, g_i\,(q_i)$$

(P10)

(10.1) subject to $\sum_i q_i = D.$

Consequently, Theorem 2 applies to P10 and, conceptually at least, separability between the production and location aspects of the problem can be attained.

The form of (10.0) suggests little about the effect of changes in the output level on the optimal location. In general, the locations will change with a change in the quantity demanded since not only will each plant's output change, but the proportion of the demand produced by individual plants may change as well. It the production functions all exhibit constant returns to scale, then (10.0) can be rewritten:

(10.2) $$\min_{q_i} \sum_i q_i (\min_{x_i} r\ d(x_i,\ b) + h_i(x_i))$$

which involves the solution of a series of single plant location problems, one for each plant, and then selection of the single plant for which $d(x_i^*,\ b) + h_i(x_i^*)$ is minimum to supply all of D.

Next consider the case of nonlinear (with quantity) transport costs for the distribution of product. Now P10 becomes:

(11.0) $$\min_{x_i,\ q_i} \sum_i C_i(q_i)\ d(x_i,\ b) + \sum_i h_i(x_i)\ g_i(q_i)$$

(P11)

(11.1) subject to $\sum_i q_i = D.$

There is little that can be said about P11 in general. A simplification can be obtained in the case where $C_i(q_i) = g_i(q_i)$ wherein (11.0) becomes:

(11.2) $$\min_{q_i} \sum_i g_i\ (q_i)\ (\min_{x_i} d(x_i,\ b) + h_i(x_i)).$$

The portion of (11.2) within parenthesis is a problem in the location variables, x_i, only, and consequently the location problem is separable from the allocation and production problems. The function $C_i(q_i)$ represents economies or diseconomies of scale (with respect to quantity) in transportation; while $g_i(q_i)$ represents a measure of economies of scale in production. When these economies of scale are the same, (11.0) can be written as (11.2). $g_i(q_i)$ has inverse relationship with the usual interpretation of economies of scale in production. For example, let $f(z) = F(H(z)) = (H(z))^\beta$ with $\beta > 1$. Then, $g_i(q_i) = F^{-1}(H(z)) = q_i^{1/\beta}$, so that $d^2\ g_i(q_i)/d\ q_i^2 < 0$. Consequently, the assumption $C_i(q_i) = g_i(q_i)$ will hold if the economies of scale in production are just "cancelled out" by diseconomies of scale in the transportation of output or vice versa.

THE MULTIPLE MARKET CASE

Now, consider a problem analogous to P9 for the case of multiple markets. The location and input mix of each plant must be determined, as well as the proportion of the known total demand at each market that will be satisfied by each plant. The problem is:

(12.0) $$\min_{x_i,\ z^i,\ q_{im}} \sum_i \sum_m r\ q_{im}\ d(x_i,\ b_m) + \sum_i \sum_j (r_j\ d(x_i,\ a_j) + p_j)\ z_j^i$$

(P12)

$$\text{subject to } \sum_m q_{im} = f_i(z^i),\ i = 1, \ldots, I \tag{12.1}$$

$$\sum_i q_{im} = q_m \quad m = 1, \ldots, M. \tag{12.2}$$

Once again utilizing Lemma 1, (12.0) can be rewritten as:

$$\min_{x_i,\, q_{im}} \sum_i \sum_m r\ q_{im}\ d(x_i,\ b_m) + \sum_i h_i\ (x_i)\ g_i\ (q_i). \tag{12.3}$$

This is a location-allocation problem in which the costs of production are incorporated in the objective function. In fact, the allocation portion is similar to a transportation problem. Thus, a rather general production-location allocation problem has been reduced through the homotheticity assumption and Lemma 1 to a form of the location-transportation problem. Solution algorithms exist for such problems, at least when the $h_i(x_i)$ and $g_i(q_i)$ functions are linear.

CONCLUDING COMMENTS

The results presented in this paper demonstrate that including consideration of input substitutability through the use of production functions often does not significantly increase the complexities encountered in solving for the optimal plant location(s), optimal input mix, and optimal allocation of product to customer. The location problem to be solved is more complex than the location problem which results when the potential for input substitution is ignored. However, algorithms similar to those developed for traditional location problems can be employed. The results derived in this paper depend upon the assumption that the production function summarizing the appropriate technology can be reasonably approximated by a homothetic function. Many theoretical or engineering based production functions and most empirical production functions can be adequately represented by homothetic functions.*

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*The derivation and use of production functions from engineering relationships has been considered in [2, 11, 15, 16, 18, 20, 21] among others.

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ECONOMETRIC FORECASTING VIA DISCOUNTED LEAST SQUARES

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ABSTRACT

Simple direct smoothing formulas are derived for updating coefficient estimates and forecasts in a discounted least squares model. These formulas are the natural extensions of R. G. Brown's well-known smoothing formulas to a general econometric setting with arbitrary explanatory time series. The recursive updating process and its forecast error properties are illustrated via a simple, yet realistic numerical example.

1. INTRODUCTION

Discounted least squares (DLS) is a generalization of the familiar ordinary least squares (OLS) model of econometrics. DLS retains the basic OLS assumption of uncorrelated homoscedastic disturbances, but it introduces artificial heteroscedasticity by discounting past squared deviations in the least squares minimization. In effect, the procedure simply gives more weight to the most recent observations. OLS is a special case of DLS where the discount factor is equal to one.

Despite the ostensible artificiality of DLS, the technique actually provides considerable flexibility in model (coefficient) updating. The discount factor can be considered as a convenient lever to adjust the dynamism of the updating process. As a practical matter, the technique works well in some real-world, short-term forecasting applications, in the sense that it improves on OLS and compares favorably with other generalized techniques.

Although DLS doesn't seem to have generated much interest among econometricians, it does form the basis for exponential smoothing models. These smoothing models relate a dependent series to specified regular functions of time, and they yield neat, direct steady-state formulas for simultaneous coefficient updating and origin translating. This steady-state updating technique was originated by Brown [1]; a more modern, accessible reference is Montgomery and Johnson [3]. Brown's specialized version of DLS is widely known and used, but to our knowledge his approach hasn't been extended explicitly to a more general setting with arbitrary explanatory time series, although the concept of recursive updating is not new (see Odell and Lewis [4]). The objective of this paper is to explicitly extend Brown's formulas to a general econometric setting and to illustrate their potential practical value via a simple, yet realistic example.

DLS is, of course, a special case of generalized least squares (GLS), and hence the solution of a DLS model is conceptually routine, in that the normal equations are straightforwardly constructed and solved via matrix inversion. Unlike arbitrary GLS models, however, the DLS normal equations can be reduced to a simple direct form for coefficient updating, and matrix inversion can be avoided after initialization. We derive these direct "smoothing" formulas in Section 3 after reviewing the general DLS structure in Section 2. Section 4 reviews Brown's specialized version of DLS and provides a conceptual link to his steady-state smoothing formulas. In Section 5, we illustrate the use of DLS in a simple econometric model; although the particular model is somewhat crude, it provides us with a device to benchmark DLS forecast error performance. Section 6 elaborates on the key issue of selecting a specific value for the discount factor.

2. THE DISCOUNTED LEAST SQUARES MODEL

Let $y(1), \ldots, y(t)$ be observations of a time series for periods $1, \ldots, t$; form the t-dimensional column vector $\mathbf{y}(t) = (y(1), \ldots, y(t))'$. Let $\mathbf{x}(1), \ldots, \mathbf{x}(t)$ be a time series of explanatory n-dimensional column vectors for periods $1, \ldots t$; form the $t \times n$ matrix $\mathbf{X}(t) = [\mathbf{x}(1), \ldots, \mathbf{x}(t)]'$. Then, we have the usual linear model formulation

$$\mathbf{y}(t) = \mathbf{X}(t)\boldsymbol{\beta} + \mathbf{u}(t) \tag{1}$$

where $\mathbf{u}(t) = (u(1), \ldots, u(t))'$ is the t-dimensional disturbance vector and $\boldsymbol{\beta}$ is the unknown n-dimensional coefficient vector. The coefficient vector $\boldsymbol{\beta}$ is assumed to be time invariant.

As usual, we assume

$$E(\mathbf{u}(t)) = \mathbf{0}$$
$$E(\mathbf{u}(t)\mathbf{u}(t)') = \sigma^2\mathbf{I}(t) \tag{2}$$

where σ^2 is an unknown scalar, $\mathbf{0}$ is the t-dimensional zero column vector, and $\mathbf{I}(t)$ is the $t \times t$ identity matrix. Rather than forming the usual (OLS) normal equations, however, we choose to minimize the discounted sum-of-squares $(\mathbf{y}(t) - \mathbf{X}(t)\mathbf{b}(t))'\mathbf{W}(t)(\mathbf{y}(t) - \mathbf{X}(t)\mathbf{b}(t))$ where $\mathbf{W}(t) = \text{diag}\{\delta^{t-1}, \ldots, \delta, 1\}$ is a diagonal $t \times t$ weight matrix incorporating the discount factor $\delta \in (0, 1]$. This minimization yields the DLS normal equations

$$\mathbf{X}(t)'\mathbf{W}(t)\mathbf{X}(t)\mathbf{b}(t) = \mathbf{X}(t)'\mathbf{W}(t)\mathbf{y}(t) \tag{3}$$

where $\mathbf{b}(t)$ is the DLS estimate of $\boldsymbol{\beta}$ at period t. Equation (3) can be written equivalently

$$\mathbf{b}(t) = \mathbf{S}(t)^{-1}\mathbf{v}(t) \tag{4}$$

where $\mathbf{v}(t) = \mathbf{X}(t)'\mathbf{W}(t)\mathbf{y}(t)$ and $\mathbf{S}(t) = \mathbf{X}(t)'\mathbf{W}(t)\mathbf{X}(t)$ is assumed nonsingular.

The effect of the weight matrix in the discounted sum-of-squares minimization is to put more emphasis on the most recent residuals. What we are doing is compensating for deficiencies in the static linear model (1); i.e., we don't really expect $\boldsymbol{\beta}$ to be time invariant so we are placing greater weight on the most recent observations.

Given that we have solved the normal equations (4) for $\mathbf{b}(t)$, we denote our forecast of $y(t + k)$, the observation k periods hence, by

$$\hat{y}_k(t + k) = \mathbf{x}(t + k)'\mathbf{b}(t) \tag{5}$$

assuming $\mathbf{x}(t+k)$ is known; in most econometric situations, of course, $\mathbf{x}(t+k)$ is itself a forecast derived by some other means. Equation (5) can be written equivalently

$$\hat{y}_k(t) = \mathbf{x}(t)'\mathbf{b}(t-k) \tag{6}$$

for $t > k$, and we define the k-period forecast error by

$$e_k(t) = y(t) - \hat{y}_k(t). \tag{7}$$

We will also refer in Sections 5 and 6 to the absolute error $a_k(t) = |e_k(t)|$ and the absolute percentage error $p_k(t) = 100\, a_k(t)/\hat{y}_k(t)$.

This section has outlined the basic assumptions and mechanics of DLS. From here on, our focus becomes computational. Our goal is to put the elements of equation (4) into a recursive format, which is why we have explicitly attached the time index to all of our variables.

3. RECURSIVE SMOOTHING FORMULAS

To develop recursive formulas, we first partition $\mathbf{X}(t)$, $\mathbf{y}(t)$, and $\mathbf{W}(t)$ as follows.

$$\mathbf{X}(t)' = [\mathbf{X}(t-1)',\ \mathbf{x}(t)]$$

$$\mathbf{y}(t)' = [\mathbf{y}(t-1)',\ y(t)]$$

$$\mathbf{W}(t) = \begin{bmatrix} \delta\mathbf{W}(t-1), & \mathbf{0} \\ \mathbf{0}', & 1 \end{bmatrix}$$

where the variables indexed by $t-1$ are defined as in Section 2 over the first $t-1$ observations and $\mathbf{0}$ is the $(t-1)$-dimensional zero column vector.

Ordinary matrix multiplication yields the following recursive formulas for $\mathbf{S}(t)$ and $\mathbf{v}(t)$.

$$\begin{aligned} \mathbf{S}(t) &= \mathbf{X}(t)'\mathbf{W}(t)\mathbf{X}(t) \\ &= \delta\mathbf{X}(t-1)'\mathbf{W}(t-1)\mathbf{X}(t-1) + \mathbf{x}(t)\mathbf{x}(t)' \\ &= \delta\mathbf{S}(t-1) + \mathbf{x}(t)\mathbf{x}(t)' \end{aligned} \tag{8}$$

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{X}(t)'\mathbf{W}(t)\mathbf{y}(t) \\ &= \delta\mathbf{X}(t-1)'\mathbf{W}(t-1)\mathbf{y}(t-1) + \mathbf{x}(t)y(t) \\ &= \delta\mathbf{v}(t-1) + \mathbf{x}(t)y(t). \end{aligned} \tag{9}$$

Equations (8) and (9) are the basic smoothing formulas for the elements of (4). However, the smoothing is still *indirect* and calculation of the coefficient estimate vector $\mathbf{b}(t)$ in (4) still requires matrix inversion at each update.

To obtain a *direct* smoothing formula, we note that

$$\begin{aligned} (\delta\mathbf{S}(t-1) + \mathbf{x}(t)\mathbf{x}(t)')\mathbf{b}(t) &= \delta\mathbf{v}(t-1) + \mathbf{x}(t)y(t) \\ &= \delta\mathbf{S}(t-1)\mathbf{b}(t-1) + \mathbf{x}(t)\,(e_1(t) + \mathbf{x}(t)'\mathbf{b}(t-1)) \end{aligned}$$

and upon rearranging terms that

$$\begin{aligned} \mathbf{S}(t)(\mathbf{b}(t) - \mathbf{b}(t-1)) &= \mathbf{x}(t)e_1(t) \\ \mathbf{b}(t) &= \mathbf{b}(t-1) + \mathbf{z}(t)e_1(t) \end{aligned} \tag{10}$$

where $\mathbf{z}(t) = \mathbf{S}(t)^{-1}\mathbf{x}(t)$. Equation (10) is a direct smoothing formula for updating $\mathbf{b}(t)$, although the calculation still involves matrix inversion at every stage.

Now (following Rao [5], problem 2.8, p. 33),

$$\mathbf{S}(t)^{-1}(\delta\mathbf{S}(t-1) + \mathbf{x}(t)\mathbf{x}(t)') = \mathbf{I}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. Hence,

$$\begin{aligned}\delta\mathbf{S}(t)^{-1}\mathbf{S}(t-1) &= \mathbf{I} - \mathbf{S}(t)^{-1}\mathbf{x}(t)\mathbf{x}(t)' \\ &= \mathbf{I} - \mathbf{z}(t)\mathbf{x}(t)'\end{aligned}$$

and upon post-multiplying each side by $\mathbf{S}(t-1)^{-1}$, we get

$$\mathbf{S}(t)^{-1} = \delta^{-1}(\mathbf{I} - \mathbf{z}(t)\mathbf{x}(t)')\mathbf{S}(t-1)^{-1} \tag{11}$$

so that $\mathbf{S}(t)^{-1}$ is obtainable directly from $\mathbf{S}(t-1)^{-1}$ via simple matrix multiplication once $\mathbf{z}(t)$ is known. To get a formula for $\mathbf{z}(t)$ in terms of $\mathbf{S}(t-1)^{-1}$, we post-multiply both sides of (11) by $\mathbf{x}(t)$ to obtain

$$\mathbf{z}(t) = \delta^{-1}(\mathbf{I} - \mathbf{z}(t)\mathbf{x}(t)')\mathbf{S}(t-1)^{-1}\mathbf{x}(t)$$

and upon rearranging terms we get

$$\mathbf{z}(t) = (\delta + \mathbf{x}(t)'\mathbf{S}(t-1)^{-1}\mathbf{x}(t))^{-1}\mathbf{S}(t-1)^{-1}\mathbf{x}(t). \tag{12}$$

To summarize, once we have obtained $\mathbf{S}(t_0)^{-1}$ and $\mathbf{b}(t_0)$ for some base period t_0 from (4), updates can be performed directly, without matrix inversion, via formulas (10), (11) and (12) together.

In this era of rapid, economical computation, avoidance of matrix inversion is not so important as it once was. Furthermore, single-equation econometric models usually don't involve large numbers of explicit coefficients because multicollinearity tends to render the estimation process unstable and meaningless; and, of course, multicollinearity presents the same problem in DLS as it does in OLS. Nevertheless, the simple, direct updating process embodied in (10), (11), and (12) is appealing and very easy to implement, especially in a matrix-oriented computer language like APL. The updating process is also very economical in terms of data storage requirements. Furthermore, equation (10) provides a conceptual bridge to the well-known steady-state smoothing formulas of R.G. Brown [1].

4. RELATIONSHIP TO BROWN'S STEADY-STATE FORMULAS

Brown [1] worked out steady-state results for the special case where $\mathbf{x}(t) = \mathbf{A}^t\mathbf{x}(0)$ for some fixed $n \times n$ transition matrix $\mathbf{A}$ and suitable starting vector $\mathbf{x}(0)$; see also Montgomery and Johnson [3] for a complete derivation. This special case of DLS includes all independent variables which are regular functions of time (e.g., polynomial and trigonometric functions of time), and it allows reduction of the direct smoothing equations to a very neat steady-state form. On the other hand, it is clear that general econometric modeling is not encompassed by this simplified framework.

Brown's special case can, of course, be treated within our general setup of Section 3, but there is no apparent simplification of the results or saving of update effort. Brown, however, obtained simplified results via an "origin shift" of the coefficients

$$\dot{\mathbf{b}}(t) = (\mathbf{A}')\mathfrak{b}(t) = (\mathbf{A}^t)\mathfrak{b}(t). \tag{13}$$

Then, since $(\mathbf{A}')^{-t} = (\mathbf{A}^{-t})'$, we get

$$\hat{y}_k(t+k) = \mathbf{x}(t+k)'\dot{\mathbf{b}}(t) = \mathbf{x}(t+k)'(\mathbf{A}')^{-t}\dot{\mathbf{b}}(t) \tag{14}$$
$$= (\mathbf{A}^{-t}\mathbf{x}(t+k))'\dot{\mathbf{b}}(t) = \mathbf{x}(k)'\dot{\mathbf{b}}(t)$$

so that forecasts depend only on the transformed coefficient estimate vector $\dot{\mathbf{b}}(t)$ and the "shifted" explanatory vector $\mathbf{x}(k)$.

The direct smoothing equation (10) becomes

$$(\mathbf{A}')^{-t}\dot{\mathbf{b}}(t) = (\mathbf{A}')^{-(t-1)}\dot{\mathbf{b}}(t-1) + \mathbf{z}(t)e_1(t). \tag{15}$$

$$\dot{\mathbf{b}}(t) = \mathbf{A}'\dot{\mathbf{b}}(t-1) + \dot{\mathbf{z}}(t)e_1(t)$$

where $\dot{\mathbf{z}}(t) = (\mathbf{A}')^t\mathbf{z}(t) = (\mathbf{A}^t)'\mathbf{z}(t)$. Now,

$$\dot{\mathbf{z}}(t) = (\mathbf{A}')^t\mathbf{S}(t)^{-1}\mathbf{x}(t) = (\mathbf{A}')^t\mathbf{S}(t)^{-1}\mathbf{A}^t\mathbf{x}(0) \tag{16}$$

$$= \dot{\mathbf{S}}(t)^{-1}\mathbf{x}(0)$$

$$\dot{\mathbf{S}}(t) = \mathbf{A}^{-t}\mathbf{S}(t)\,(\mathbf{A}')^{-t} = \mathbf{A}^{-t}\mathbf{S}(t)\,(\mathbf{A}^{-t})' \tag{17}$$
$$= \mathbf{A}^{-t}\left(\sum_{j=1}^{t}\delta^{t-j}(\mathbf{A}^j\mathbf{x}(0))\,(\mathbf{A}^j\mathbf{x}(0))'\right)(\mathbf{A}^{-t})'$$
$$= \sum_{j=1}^{t}\delta^{t-j}(\mathbf{A}^{-(t-j)}\mathbf{x}(0))\,(\mathbf{A}^{-(t-j)}\mathbf{x}(0))'$$
$$= \sum_{j=0}^{t-1}\delta^j(\mathbf{A}^{-j}\mathbf{x}(0))\,(\mathbf{A}^{-j}\mathbf{x}(0))'.$$

For a well-behaved transition matrix $\mathbf{A}$ and $\delta \in (0, 1)$, the limit $\dot{\mathbf{S}}(\infty) = \lim_{t\to\infty}\dot{\mathbf{S}}(t)$ exists and is generally approached rapidly. Hence, equation (15) rapidly takes the steady-state form

$$\dot{\mathbf{b}}(t) = \mathbf{A}'\dot{\mathbf{b}}(t-1) + \dot{\mathbf{z}}(\infty)e_1(t) \tag{18}$$

where $\dot{\mathbf{z}}(\infty) = \dot{\mathbf{S}}(\infty)^{-1}\mathbf{x}(0)$. This is a very neat result in that the matrix $\dot{\mathbf{S}}(\infty)$ is easily computed for a given discount factor and need only be inverted once to get the steady-state multiplier $\dot{\mathbf{z}}(\infty)$; then, given a starting vector $\dot{\mathbf{b}}(0)$, steady-state updates can occur routinely via equation (18). In practice, an initial, untransformed DLS estimate $\mathbf{b}(t_0)$ is obtained via equation (4) and $\dot{\mathbf{b}}(t_0) = (\mathbf{A}')^{t_0}\mathbf{b}(t_0)$ is used as $\dot{\mathbf{b}}(0)$ to start the steady-state update process (18); otherwise, the starting vector $\dot{\mathbf{b}}(0)$ is chosen judgmentally.

Besides its mathematical simplicity, the steady-state update process (18) has the additional desirable feature of very limited data storage requirements. Once computed, $\dot{\mathbf{z}}(\infty)$ is a fixed parameter vector and $\mathbf{A}$ is a fixed parameter matrix; besides these fixed parameters, one need only maintain the most recently updated coefficient estimate vector $\dot{\mathbf{b}}(t)$. It is worth noting that the generalized recursive procedure of Section 3 is just as parsimonious in terms of data storage requirements, although more routine updating is necessary (e.g., $\mathbf{S}(t)^{-1}$ and $\mathbf{z}(t)$, as well as $\mathbf{b}(t)$). The following two examples illustrate both Brown's steady-state updating process and the more general recursive updating process of Section 3.

EXAMPLE: SIMPLE EXPONENTIAL SMOOTHING

In this example, we assume $E(y(t)) = \beta$ for all t, where β is an unknown scalar constant. Hence, $n = 1$ and we have the simple scalar relationships $x(t) = A = 1$, $\dot{b}(t) = b(t)$, $\dot{S}(t) = S(t)$, and $\dot{z}(t) = z(t)$ for all t. Assuming $\delta \in (0, 1)$ and $\alpha = 1 - \delta$, we have

$$S(t) = \sum_{j=1}^{t} \delta^{t-j} = \sum_{j=0}^{t-1} \delta^j = (1 - \delta^t)\alpha^{-1} \to \alpha^{-1}$$

$$z(t) = S(t)^{-1} \to \alpha.$$

Hence, the transient update equation ((10) or (15)) takes the form

$$b(t) = b(t-1) + (1 - \delta^t)^{-1}\alpha e_1(t)$$

while the steady-state update equation (18) takes the familiar form

$$b(t) = b(t-1) + \alpha e_1(t).$$

In this simple situation $\hat{y}_k(t + k) = b(t)$ for all $k \geqslant 1$, so our forecasts for all future periods at time t are identical.

EXAMPLE: EXPONENTIAL SMOOTHING WITH TIME TREND

In this example, we assume

$$E(y(t)) = \beta_1 + \beta_2 t = \mathbf{x}(t)'\boldsymbol{\beta}$$

for all t, where $n = 2$, $\boldsymbol{\beta} = (\beta_1, \beta_2)'$ is an unknown vector consisting of intercept β_1 and trend (or slope) β_2, and $\mathbf{x}(t) = (1, t)'$. In Brown's setup, $\mathbf{x}(t) = \mathbf{A}^t\mathbf{x}(0)$ where $\mathbf{x}(0) = (1, 0)'$ and

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

It follows that

$$\mathbf{A}^j = \begin{bmatrix} 1 & 0 \\ j & 1 \end{bmatrix}$$

and $\mathbf{A}^j\mathbf{x}(0) = (1, j)'$ for any real integer j. Assuming $\delta \in (0, 1)$ and $\alpha = 1 - \delta$, we get

$$\dot{\mathbf{S}}(t) = \sum_{j=0}^{t-1} \delta^j \begin{bmatrix} 1 & -j \\ -j & j^2 \end{bmatrix} = \begin{bmatrix} f_0(t) & -f_1(t) \\ -f_1(t) & f_2(t) \end{bmatrix}$$

where

$$f_0(t) = \sum_{j=0}^{t-1} \delta^j = (1 - \delta^t)\alpha^{-1} \to \alpha^{-1}$$

$$f_1(t) = \sum_{j=0}^{t-1} j\delta^j = (\delta f_0(t) - t\delta^t)\alpha^{-1} \to \delta\alpha^{-2}$$

$$f_2(t) = \sum_{j=0}^{t-1} j^2\delta^j = ((1 + \delta)f_1(t) - t(t-1)\delta^t)\alpha^{-1} \to \delta(1 + \delta)\alpha^{-3}.$$

Then,

$$\dot{\mathbf{z}}(t) = \dot{\mathbf{S}}(t)^{-1}\mathbf{x}(0)$$
$$= (f_0(t)f_2(t) - f_1(t)^2)^{-1}(f_2(t), f_1(t))' \to (\alpha(1+\delta), \alpha^2)'.$$

Hence, the transient update equation (15) takes the form

$$\dot{b}_1(t) = \dot{b}_1(t-1) + \dot{b}_2(t-1) + \dot{z}_1(t)e_1(t)$$
$$\dot{b}_2(t) = \dot{b}_2(t-1) + \dot{z}_2(t)e_1(t)$$

where

$$\dot{z}_1(t) = (f_0(t)f_2(t) - f_1(t)^2)^{-1}f_2(t)$$
$$\dot{z}_2(t) = (f_0(t)f_2(t) - f_1(t)^2)^{-1}f_1(t)$$

and the steady-state update equation (18) takes the familiar form

$$\dot{b}_1(t) = \dot{b}_1(t-1) + \dot{b}_2(t-1) + \alpha(1+\delta)e_1(t)$$
$$\dot{b}_2(t) = \dot{b}_2(t-1) + \alpha^2 e_1(t).$$

Since $\mathbf{z}(t) = (\mathbf{A}^{-t})'\dot{\mathbf{z}}(t) = (\dot{z}_1(t) - \dot{z}_2(t)t, \dot{z}_2(t))'$, we have $z_1(t) \simeq \dot{z}_1(\infty) - \dot{z}_2(\infty)t = \alpha(1+\delta) - \alpha^2 t$ and $z_2(t) \simeq \alpha^2$ for t sufficiently large. Hence, the general update equation (10) takes the form

$$b_1(t) = b_1(t-1) + (\dot{z}_1(t) - \dot{z}_2(t)t)e_1(t)$$
$$b_2(t) = b_2(t-1) + \dot{z}_2(t)e_1(t)$$

which indicates that the intercept update process would become increasingly volatile over time. This volatility, however, is immaterial to the forecasting process. Since $\dot{\mathbf{b}}(t) = (\mathbf{A}^t)'\mathbf{b}(t) \simeq (b_1(t) + b_2(t)t, b_2(t))'$, the forecast for the k-th future period at time t is identically

$$\hat{y}_k(t+k) = b_1(t) + b_2(t)(t+k)$$
$$= (b_1(t) + b_2(t)t) + b_2(t)k$$
$$= \dot{b}_1(t) + \dot{b}_2(t)k.$$

Brown's steady-state approach is very attractive for DLS models which depend on regular functions of time. We think the smoothing formulas of Section 3 provide a straightforward extension of Brown's approach to general econometric models with arbitrary explanatory time series. The second example above indicates that the general update process (10) is likely to be less volatile over the long run if systematic growth components are removed from the various series beforehand by first-differencing or some other means. We have not found this to be an important point in practical econometric work, however. The example in the next section illustrates a simple model where business cycle factors predominate and drive the coefficient (forecast) update process.

5. A SIMPLE ECONOMETRIC EXAMPLE

To illustrate the efficacy of DLS in an econometric context, we put $y(t)$ equal to total U.S. new passenger car unit sales in quarter t and $\mathbf{x}(t) = (x_1(t), x_2(t))'$, where $x_1(t) \equiv 1$ and $x_2(t)$ is the motor vehicles and parts personal consumption component of real Gross National Product (1972$) in quarter t. Both $y(t)$ and $x_2(t)$ are seasonally adjusted, annualized rates

from the Bureau of Economic Analysis, U.S. Department of Commerce. The overall time interval ranges from the first quarter of 1970 to the last quarter of 1980. We use the 12 quarter interval from 1970:1 to 1972:4 to initialize the model. We use the 32 quarter interval from 1973:1 to 1980:4 to monitor the update process and to measure *ex post* forecast errors for 1, 2, 3, and 4 quarters ahead. The simulated ex post forecasts incorporate actual historical observations for the consumption series $x_2(t)$; in reality, future values would have to be estimated (ex ante) from some type of quarterly GNP forecasting model with the requisite level of disaggregation.

Now clearly the time series $y(t)$ and $x_2(t)$ should be highly correlated since most new passenger car sales (aside from business and government purchases) are imbedded in the broader personal consumption series, which also includes a net used car component, small trucks, other recreational vehicles, service parts, and so forth. On the other hand, the relationship between the two series is far from exact and may change over time, cyclically and perhaps secularly. Moreover, the new car sales series is widely followed as a cyclical indicator, and a simple mechanism to generate forecasts from a GNP component is very desirable.

Table 1 summarizes the average signed, absolute, and absolute percentage forecast errors 1, 2, and 4 quarters ahead (as defined in Section 2) for discount factors ranging from .1 to 1.0; the omitted 3 quarter ahead results are analogous. Our focus on simple arithmetic error measures is intuitive. Despite the least-squares basis of DLS, we think most people conceptualize forecast errors in these simple arithmetic terms.

The average errors for a first-order autoregressive (AR1) model are also included in Table 1. This GLS model (see Johnston [2], Chapter 9, and Theil [6], Section 6.3) is commonly used in econometric work when the OLS Durbin-Watson statistic turns out exceptionally small, as it did in this case over most of the simulated ranges. The AR1 solution procedure is also commonly known as the Cochrane-Orcutt technique.

Table 2 contains detailed quarter-by-quarter coefficient adjustments and forecasts for the case $\delta = .5$. Table 3 contains the full range of input data and intermediate results for that same case $\delta = .5$. This example is computationally very simple, so we just iterated the indirect equations (8), (9), and (4) to get updated coefficient estimates $\mathbf{b}(t) = (b_1(t), b_2(t))'$ for each quarter t. Table 2 and 3 entries have been suitably rounded for presentation purposes.

It is clear from the simulation results that OLS is significantly inferior to both AR1 and alternative DLS models in terms of average absolute forecast error, and furthermore that AR1 is inferior to the best DLS models with discount factors in the .5 neighborhood. Since the DLS solution procedure is also much simpler than AR1, the approach certainly seems worthy of consideration in many short-term forecasting situations where AR1 is routinely applied.

It is interesting to note in Table 1 that DLS forecast bias generally increases with the forecast horizon for a fixed discount factor and with the discount factor for a fixed forecast horizon. On the other hand, the average absolute forecast error, in this particular example, is approximately minimized across all the forecast horizons with discount factor in the .5 neighborhood.

TABLE 1 — *Summary Forecast Error Analysis*

	Average 1973:1-1980:4			Average 1973:2-1980:4			Average 1973:4-1980:4		
δ	e_1	a_1	p_1	e_2	a_2	p_2	e_4	a_4	p_4
.1	−.09	.56	6.7	.06	.82	22.4	−.27	.86	9.3
.2	−.05	.39	4.6	−.05	.48	5.5	−.33	.57	5.7
.3	−.04	.31	3.5	−.08	.35	3.8	−.34	.51	5.1
.4	−.05	.27	2.8	−.11	.30	3.1	−.36	.49	4.9
.5	−.07	.25	2.6	−.15	.30	3.0	−.40	.49	4.8
.6	−.11	.26	2.6	−.20	.31	3.1	−.45	.50	4.9
.7	−.17	.29	3.0	−.27	.36	3.6	−.54	.58	5.7
.8	−.29	.37	3.7	−.41	.45	4.5	−.67	.68	6.7
.9	−.51	.53	5.3	−.62	.63	6.2	−.84	.84	8.1
1.0(OLS)	−.84	.84	8.1	−.92	.92	8.7	−1.09	1.09	10.2
AR1	−.19	.35	3.5	−.32	.50	5.0	−.64	.74	7.0

TABLE 2 — *Detailed Results for* $\delta = .5$

Quarter	t	$b_1(t)$	$b_2(t)$	$x_2(t)$	$y(t)$	$\hat{y}_1(t)$	$\hat{y}_2(t)$	$\hat{y}_3(t)$	$\hat{y}_4(t)$
1972:4	12	2.67	0.158	56.5	11.6	NA	NA	NA	NA
1973:1	13	2.32	0.164	60.7	12.3	12.2	NA	NA	NA
1973:2	14	2.12	0.170	58.0	12.1	11.8	11.8	NA	NA
1973.3	15	−0.58	0.213	55.6	11.1	11.6	11.5	11.4	NA
1973.4	16	−5.92	0.306	51.7	9.8	10.5	10.9	10.8	10.8
1974:1	17	−2.47	0.244	47.9	9.3	8.7	9.6	10.3	10.2
1974:2	18	−3.45	0.262	47.9	9.0	9.2	8.7	9.6	10.3
1974:3	19	−3.32	0.257	49.9	9.4	9.6	9.7	9.4	10.1
1974:4	20	−4.18	0.274	42.3	7.4	7.5	7.6	7.8	7.0
1975:1	21	−2.86	0.249	43.7	8.2	7.8	7.9	8.0	8.2
1975:2	22	−3.76	0.264	44.2	7.7	8.2	7.9	8.0	8.1
1975:3	23	−0.68	0.194	49.8	8.9	9.4	9.5	9.5	9.5
1975:4	24	−0.01	0.179	52.2	9.3	9.4	10.0	10.1	10.1
1976:1	25	0.77	0.163	56.9	10.0	10.2	10.3	11.3	11.3
1976:2	26	1.62	0.145	57.0	9.8	10.0	10.2	10.4	11.3
1976:3	27	1.79	0.142	57.4	9.9	9.9	10.1	10.3	10.4
1976:4	28	0.83	0.160	58.0	10.2	10.0	10.0	10.2	10.4
1977:1	29	0.10	0.173	63.5	11.1	11.0	10.8	10.8	11.1
1977:2	30	−0.75	0.188	63.1	11.2	11.0	10.9	10.7	10.8
1977:3	31	0.38	0.168	63.2	10.9	11.2	11.0	11.0	10.7
1977:4	32	1.71	0.146	64.1	11.0	11.2	11.3	11.2	11.1
1978:1	33	0.72	0.161	62.4	10.7	10.8	10.9	11.0	10.9
1978:2	34	0.19	0.169	68.0	11.7	11.7	11.6	11.8	12.1
1978:3	35	0.22	0.168	65.6	11.2	11.3	11.3	11.3	11.4
1978:4	36	3.27	0.119	66.8	11.1	11.4	11.5	11.5	11.5
1979:1	37	2.66	0.130	66.4	11.4	11.2	11.4	11.4	11.4
1979:2	38	1.85	0.142	59.4	10.3	10.4	10.4	10.2	10.2
1979:3	39	2.82	0.128	60.8	10.7	10.5	10.6	10.5	10.4
1979:4	40	−1.71	0.196	60.3	9.8	10.6	10.4	10.5	10.5
1980:1	41	−2.48	0.210	62.1	10.6	10.5	10.8	10.7	10.7
1980:2	42	−1.98	0.202	47.0	7.5	7.4	7.5	8.8	8.5
1980:3	43	−1.28	0.192	51.5	8.8	8.4	8.3	8.4	9.4
1980:4	44	−1.17	0.189	54.6	9.1	9.2	9.0	9.0	9.0

TABLE 3 — *Input Data and Intermediate Results for* $\delta = .5$

Quarter	t	$x_2(t)$	$y(t)$	$S_{11}(t)$	$S_{12}(t)$	$S_{22}(t)$	$v_1(t)$	$v_2(t)$
1970:1	1	39.0	8.8	1.00	39.0	1,521	8.8	343
1970:2	2	40.3	9.1	1.50	59.8	2,385	13.5	538
1970:3	3	40.2	9.0	1.75	70.1	2,808	15.8	631
1970:4	4	33.1	6.9	1.88	68.1	2,500	14.8	544
1971:1	5	42.6	10.0	1.94	76.7	3,065	17.4	698
1971:2	6	44.1	9.9	1.97	82.4	3,477	18.6	786
1971:3	7	46.1	10.4	1.98	87.3	3,864	19.7	872
1971:4	8	49.3	10.6	1.99	93.0	4,362	20.4	959
1972:1	9	49.7	10.5	2.00	96.2	4,651	20.7	1,001
1972:2	10	51.2	10.7	2.00	99.3	4,947	21.1	1,048
1972:3	11	52.2	10.8	2.00	101.8	5,198	21.3	1,088
1972:4	12	56.5	11.6	2.00	107.4	5,791	22.3	1,199
1973:1	13	60.7	12.3	2.00	114.4	6,580	23.4	1,346
1973:2	14	58.0	12.1	2.00	115.2	6,654	23.8	1,375
1973:3	15	55.6	11.1	2.00	113.2	6,418	23.0	1,305
1973:4	16	51.7	9.8	2.00	108.3	5,882	21.3	1,159
1974:1	17	47.9	9.3	2.00	102.1	5,235	20.0	1,025
1974:2	18	47.9	9.0	2.00	98.9	4,912	19.0	944
1974:3	19	49.9	9.4	2.00	99.4	4,946	18.9	941
1974:4	20	42.3	7.4	2.00	92.0	4,262	16.8	783
1975:1	21	43.7	8.2	2.00	89.7	4,041	16.6	750
1975:2	22	44.2	7.7	2.00	89.0	3,974	16.0	715
1975:3	23	49.8	8.9	2.00	94.3	4,467	16.9	801
1975:4	24	52.2	9.3	2.00	99.4	4,958	17.8	886
1976:1	25	56.9	10.0	2.00	106.6	5,717	18.9	1,012
1976:2	26	57.0	9.8	2.00	110.3	6,107	19.2	1,065
1976:3	27	57.4	9.9	2.00	112.5	6,348	19.5	1,101
1976:4	28	58.0	10.2	2.00	114.3	6,538	20.0	1,142
1977:1	29	63.5	11.1	2.00	120.6	7,301	21.1	1,276
1977:2	30	63.1	11.2	2.00	123.4	7,632	21.7	1,345
1977:3	31	63.2	10.9	2.00	124.9	7,810	21.8	1,361
1977:4	32	64.1	11.0	2.00	126.6	8,014	21.9	1,386
1978:1	33	62.4	10.7	2.00	125.7	7,901	21.6	1,361
1978:2	34	68.0	11.7	2.00	130.8	8,574	22.5	1,476
1978:3	35	65.6	11.2	2.00	131.0	8,591	22.5	1,473
1978:4	36	66.8	11.1	2.00	132.3	8,758	22.3	1,478
1979:1	37	66.4	11.4	2.00	132.6	8,788	22.6	1,496
1979:2	38	59.4	10.3	2.00	125.7	7,922	21.6	1,360
1979:3	39	60.8	10.7	2.00	123.6	7,658	21.5	1,330
1979:4	40	60.3	9.8	2.00	122.1	7,465	20.5	1,256
1980:1	41	62.1	10.6	2.00	123.2	7,589	20.9	1,286
1980:2	42	47.0	7.5	2.00	108.6	6,003	17.9	996
1980:3	43	51.5	8.8	2.00	105.8	5,654	17.8	951
1980:4	44	54.6	9.1	2.00	107.5	5,808	18.0	972

Moreover, there is little difference in performance for discount factors in the fairly broad range .3 to .7, indicating robustness with respect to choice of this judgmental parameter. For very small discount factors, the coefficients evidently adjust rapidly to eliminate bias but the resulting procedure is excessively volatile. For very large discount factors (e.g., OLS), the coefficients adjust slowly and the resulting procedure, although smoother, is quite biased. For mid-range discount factors (around .5 in this example), the resulting procedure adjusts rapidly enough to keep bias small while also providing sufficient stability to yield relatively small average absolute errors; i.e., it is a good blend of dynamism and stability.

Table 2 shows how the coefficients and forecasts fluctuated over the simulation interval 1973:1 to 1980:4 for the quasi-optimum case $\delta = .5$. Note that the slope estimate $b_2(t)$ is far more stable than the intercept estimate $b_1(t)$. Note also the characteristic pattern of negative intercepts during the 1973-75 recessionary contraction, a pattern commencing once again in late 1979. The forecasts are, for the most part, pretty close to the actual new car sales.

This crude forecasting model can probably be improved econometrically. For example, we have ignored the possible incremental impact of strikes and interest rate fluctuations on new car sales. We have also made no attempt to differentiate domestics from imports. Furthermore, we are well aware that, as a practical matter, bias is often reduced in OLS models by adjusting the forecasts with judgmental "add factors". The purpose of this example is merely to indicate the potential value of DLS.

6. CHOOSING THE DISCOUNT FACTOR

Whenever DLS is employed in any form, choice of the discount factor becomes an important issue. In the example of Section 5, ex post simulation showed $\delta = .5$ to be a good choice. This is not always the case, however. We've analyzed other cases where greater inherent volatility in the dependent series caused a larger discount factor to be approximately optimum. In any event, this kind of ex post simulation analysis is obviously a good idea when plenty of historical data exists. Otherwise, one must resort to judgmental criteria.

From a judgmental standpoint, the author would tend to focus on discount factors between .3 and 1.0 (OLS) to maintain some degree of stability in the coefficient update process. If OLS on the historical data yields a very low Durbin-Watson statistic, one would naturally choose a lower discount factor in this range to enhance the dynamism of the process. In this case, DLS provides an alternative to the ubiquitous AR1 procedure.

Another judgmental criterion is the "moving average equivalence" principle (see Montgomery and Johnson [3], p. 52). The idea is to equate (very approximately) a DLS model with an equivalent "rolling" OLS model, over the most recent m historical periods, in terms of the "average age of the data" $\sum_{j=0}^{m-1} j/m = (m-1)/2$. Based on simple exponential smoothing, we get the equivalence relationship $\delta = (m-1)/(m+1)$ or $m = (1+\delta)/(1-\delta)$; this equivalence is fairly crude when extended to general DLS models, but it is nonetheless appealing. For example, if we wanted the average age of our data to be 3 periods, we would choose $\delta = .5$; for 4 periods, $\delta = .6$; and for 9 periods, $\delta = .8$. In many short-term forecasting problems with quarterly data, an average age of 2 to 8 quarters seems intuitively appealing; this corresponds to $.33 \leqslant \delta \leqslant .78$.

Another possibility is to adjust the discount factor itself over time. Trigg and Leach [7] introduced an intuitive technique to adjust the discount factor in response to a buildup of systematic forecast errors (see also Montgomery and Johnson [3], pp. 175-179). Specifically, they make δ a decreasing function of $|\bar{e}_1(t)|/\bar{a}_1(t)$, where $\bar{e}_1(t)$ and $\bar{a}_1(t)$ are simple exponentially smoothed values of the one-period-ahead signed and absolute forecast errors respectively; this adjustment decreases δ to increase responsiveness when there is a buildup of signed forecast error (or bias) in relation to the mean absolute forecast error. The Trigg-Leach adaptive technique may induce instability into some DLS models, and we haven't tested it in a general context. Nevertheless, the technique is widely employed as an enhancement to simple exponential smoothing, its volatility is easily constrained, and the general approach seems like an avenue worth exploring in practical forecasting situations.

As we've seen, there is no hard and fast rule for choosing a discount factor, although it appears that good values can be estimated in specific situations and there are even possibilities for adjusting the factor adaptively. The fact that DLS incorporates a judgmental factor should not be considered a negative. The discount factor merely adds practical flexibility.

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SPARES STOCKING POLICIES FOR REPAIRABLE ITEMS WITH DEPENDENT REPAIR TIMES

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ABSTRACT

This paper analyzes the problem of determining desirable spares inventory levels for repairable items with dependent repair times. The problem is important for repairable products such as aircraft engines which can have very large investment in spares inventory levels. While existing models can be used to determine optimal inventory spares levels when repair times are independent, the practical considerations of limited repair shop capacity and prioritized shop dispatching rules combine to make repair times not independent of one another. In this research a simulation model of a limited capacity repair facility with prioritized scheduling is used to explore a variety of heuristic approaches to the spares stocking decision. The heuristics are also compared with use of a model requiring independent repair times (even though that assumption is not valid here). The results show that even when repair time dependencies are present, the performance of a model which assumes independent repair times is quite good.

1. INTRODUCTION

This paper focuses on the problem of determining a spares stocking policy for a repairable item inventory system with dependent repair times. If repair times are assumed to be independent, there are several procedures available for determining an optimal spares stocking policy (see [2], [5-7], [13]). However, when repair times are dependent, as they are when there is limited or shared repair capacity and items are subject to priority scheduling, existing theory is no longer strictly applicable.

Repairable inventory systems are systems in which failed units are repaired and returned to service (rather than scrapped and replaced by new items). One important example of such a system is the rework and repair process for jet engines for commercial and military aircraft. A modern jet engine has a hierarchical (indentured) product structure; it is composed of several

subassemblies called modules, which in turn are composed of a number of parts called components. An engine "failure" or need for maintenance is caused by the malfunction of one or more modules, which are in turn caused by one or more component failures. This hierarchical product structure has three indenture levels—the engine, the modules and the components.

Consider a commercial airline with a large number of planes in its fleet. At random times, an engine malfunction is discovered and the engine is removed from the wing for repair. If an identical spare engine is available it is used to replace the failure and the plane is returned to service. If no spares are available, the plane is grounded until a repaired engine of the same type is completed by the repair facility.

The engine which is removed is sent to the repair facility where one or more modules requiring rework are identified (see Figure 1). These modules are removed from the engine and are replaced by identical spares if they are available. If not, the engine must wait in its repair stall until the proper modules emerge from the repair process. A similar procedure is followed for the components which require rework. The component parts are repaired (reworked) in the repair shop. The repair work content of modules and engines is primarily inspection, disassembly and assembly, and testing, with little or no machine center processing required.

The demand for spare components is dependent on the repair requirements for modules; module demands are directly dependent on engine requirements. Performance of this system is a function of three major factors:

1. The *target spares inventory levels* for engines, modules and components. The function of spares inventory is to provide system support while the faulty items are repaired.

2. The *capacity to repair* components and to perform inspection, assembly, and testing on modules and engines. Larger shop capacity results in shorter average shop repair times (called cycle times), but of course, a lower percentage utilization of capacity.

3. The *priority scheduling system* used in the repair shop. A responsive scheduling system should reduce repair times for those components which are currently in short supply while allowing less critical components to be repaired more slowly. Priorities will change over time as different parts become critical.

A previous study [4] examined the effects of alternative scheduling policies on the system. In this study, the repair shop capacity and the priority scheduling policy are held constant and various procedures for determining improved spares stocking policies are investigated.

2. RELEVANT LITERATURE

One of the most important problems in a repairable inventory system is the determination of an optimal spares stocking policy, given a total spares budget. A thorough review of the research work on this problem has recently been provided by Nahmias [8]. Much of this literature is concerned not only with optimal spares levels in a hierarchical product structure, but also the allocation of these spares to various geographically separate locations (echelons) in a decentralized logistical support system.

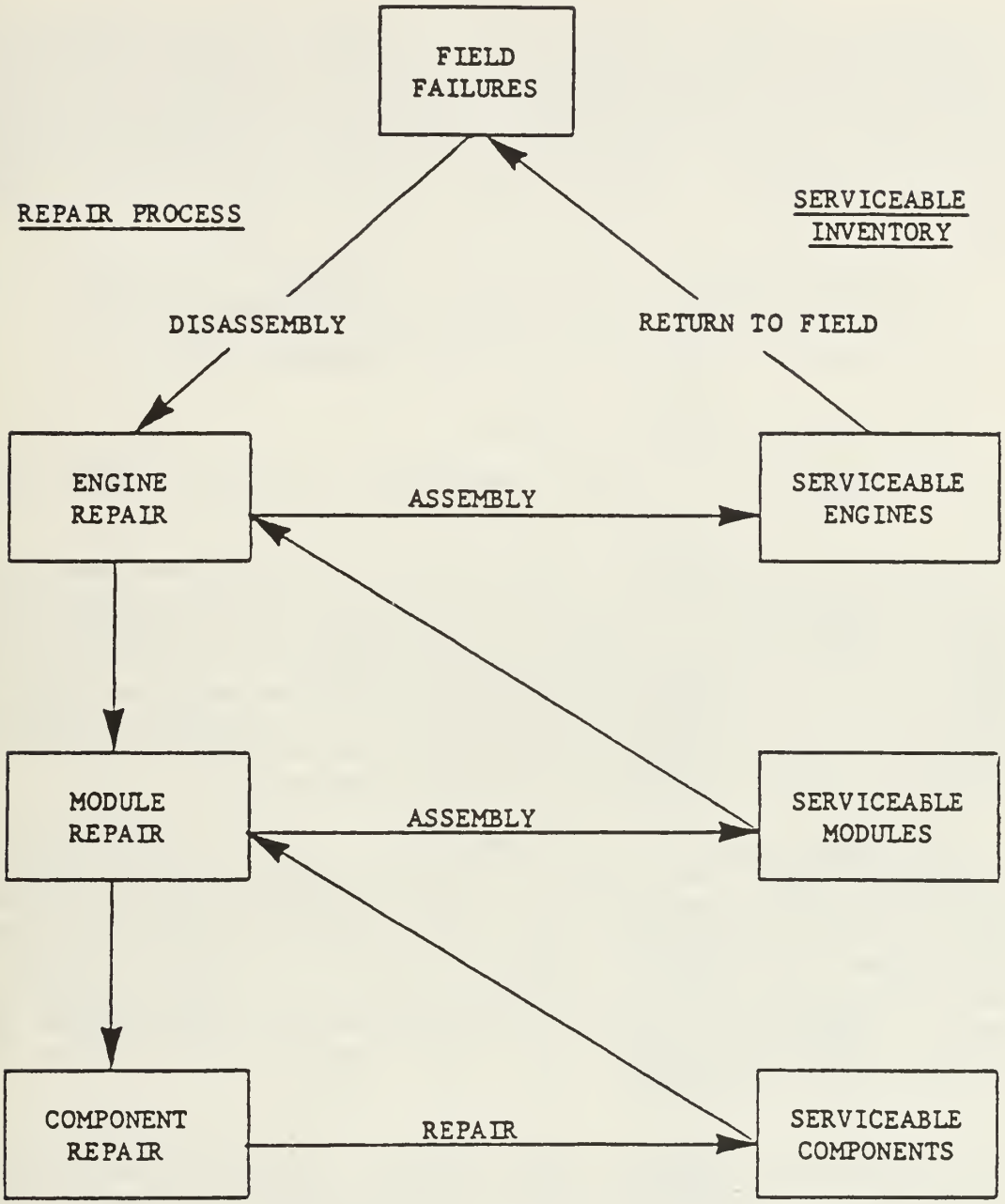

FIGURE 1. Schematic of repairable inventory flow.

The usual control system here is a continuous review, one-for-one $(S - 1, S)$ ordering policy. Thus, whenever an item fails, an order is immediately placed for a replacement part of the same type, with an order quantity of one, and the failed item is sent immediately to the repair shop (i.e., batching items for repair is not economical)

Sherbrooke's METRIC model [14] provided an important analytical treatment of the optimal spares problem. His model determines an optimal spares policy for a system consisting of a central repair depot with several subordinate bases. His objective was to minimize expected backorders for all items (components as well as engines). Muckstadt [5] subsequently noted that only backorders for engines affect the availability of the aircraft. Muckstadt's formulation, called MOD-METRIC, minimized expected backorders for the end item, i.e., engines, subject to an investment constraint on total dollars allocated to both the engine and module spares.

The following assumptions are made in Sherbrooke's and Muckstadt's models:

(1) Demands are generated by a stationary Compound Poisson process.

(2) A single module fails in each engine.

(3) Repair times for all items are independent (i.e., there is no waiting or batching before failed items begin the repair process).

The assumption of independent repair times enabled Sherbrooke and Muckstadt to utilize a queuing theory result called Palm's Theorem [9], which states that if failures are generated by a stationary Poisson process and repair times are independent, identically distributed random variables, then the steady-state number of items undergoing repair at any given time is also Poisson with a mean equal to the product of the item's failure rate and its mean repair time. These models and related research require assumption (3) of independence of repair times. For a shop with limited or shared capacity, repair times will not be independent. Therefore, these models are not directly applicable to the problem considered here.

Another set of models, the so-called "machine repair" models, can represent queuing in the repair shop. These models assume that there are m operational machines (engines), n identical spares, and $S \geqslant 1$ parallel servers or repair stations. When a failure occurs, it queues for repair. Utilizing variations of this type of model and assuming repair times are exponentially distributed, several authors (see references [1], [2], [3] and [11]) have examined the problem of how many spares and repair channels are required for a system to operate at a given service level. However, the limitation of exponential repair times is overly restrictive. Moreover, these models cannot represent dynamic priority scheduling rules such as those found most effective in a previous study [4].

In this article, a simulation model of a hypothetical repair shop will be used to analyze the spares stocking problem when repair times are dependent. First, several heuristic indices are used to find spares levels which approximate the "efficient frontier" trading off inventory investment versus engine backorders. Next, the spares budget is fixed and two alternative methods of determining spares stocking levels are investigated. Finally, the sensitivity of system performance to the distribution of spares among the three indenture levels is explored.

3. THE MODEL

The simulation model used is identical to that presented in Hausman and Scudder [4] to examine different priority scheduling rules and will not be repeated in detail here. However, it is useful here to present those elements which are of importance to this paper.

The model contains ten work centers, each with limited capacity. The model simulates actual queuing in work centers, thereby representing the dependent repair times described initially. Table 1 presents the conditional failure rates for each of the modules and components in the model. It is assumed that a single module fails in each engine and a single component fails in each module.

TABLE 1 — *Failure Probabilities (Single-Failure Model)*

Module	Module Failure Probability	Component	Conditional Component Failure Probability
1	0.1667	11	0.2903
		12	0.1936
		13	0.2903
		14	0.2258
			1.0000
2	0.0833	21	0.3333
		22	0.6000
		23	0.0667
			1.0000
3	0.2500	31	0.1026
		32	0.0256
		33	0.1538
		34	0.2308
		35	0.2051
		36	0.2308
		37	0.0513
			1.0000
4	0.2778	41	0.0800
		42	0.3200
		43	0.2000
		44	0.4000
			1.0000
5	0.2222	51	0.3200
		52	0.1600
		53	0.2400
		54	0.0800
		55	0.2000
	1.0000		1.0000

The performance criterion used is expected delay days for engines (i.e., time-weighted engine backorders). Mathematically this quantity is calculated by dividing expected engine backorders by the constant daily mean engine failure rate. Thus, an equivalent performance measure is expected engine backorders. This criterion is used as it is representative of the most critical aspect of system performance—the grounding of a plan for lack of an engine. It is also the criterion found in the majority of the literature.

The initial spares vector used in the model is obtained using a combination of enumeration and an approximate technique proposed by Muckstadt [6]. Engine and module spares levels are determined using complete enumeration of all possible combinations for a given total budget. (Although this may appear to involve a large number of iterations, the relevant allocations are easily determined due to the quasi-convexity of the expected backorder function as shown by Muckstadt [6].) For any given combination, the remaining budget is then allocated to component spares using Muckstadt's approximation technique [6].

Muckstadt's method approximates the expected backorder function by an exponential function at the lowest indenture level (i.e., components in our model). The objective is to minimize the sum of the expected backorders for all the components subject to a budget constraint (i.e., the remaining budget). Specifically, let:

$B_i(S_i) =$ expected backorders for item i given an intial stocking level S_i.

Then

$$B_i(S_i) = \sum_{x > S_i} (x - S_i)\, p(x|\lambda_i \bar{R}_i) \tag{1}$$

where

S_i = stocking level for item i
λ_i = failure rate for item i
$\bar{R}_i$ = mean repair time for item i
$p(x|\lambda_i \bar{R}_i)$ = Poisson probability with mean $\lambda_i \bar{R}_i$.

The problem then is:

$$\min \sum_{\text{all } i} B_i(S_i)$$

$$\text{subject to } \sum_{\text{all } i} C_i S_i \leqslant C$$

where

C_i = unit cost of item i
C = available budget.

The approximation made is:

$$B_i(S_i) \cong a_i e^{-b_i S_i} \tag{2}$$

where a_i and b_i are positive constants to be determined. This approximation is shown to have high accuracy when $S_i \geqslant \lambda_i \bar{R}_i$, that is, when spares for item i equal or exceed the expected quantity of item i in resupply in the system. This condition, analogous to a reorder point equaling expected lead time demand plus positive safety stock, will typically hold; i.e., we expect positive safety stock, not negative safety stock. Values for a_i and b_i are determined by calculating the actual expected backorders for various values of S_i, λ_i, and $\bar{R}_i$ using (1) and fitting the logarithm of the exponential function in (2) using linear regression. Note that this approximation is indirectly affected by the mean repair times, $\bar{R}_i$. For our purposes, the mean repair times obtained from FCFS priority scheduling are used. The spares vector produced by this procedure is shown in Table 2, together with the costs of each item.

TABLE 2 — *Initial Spares Vector*

Item	Unit Cost	Number of Spares	Total Cost
Engine	$2,000,000	0	$ 0
Module 1	400,000	2	800,000
Module 2	308,000	1	308,000
Module 3	630,000	2	1,260,000
Module 4	490,000	2	980,000
Module 5	575,000	2	1,150,000
Component 11	135,000	2	270,000
Component 12	85,000	3	255,000
Component 13	150,000	2	300,000
Component 14	110,000	3	330,000
Component 21	80,000	2	160,000
Component 22	100,000	4	400,000
Component 23	60,000	0	0
Component 31	160,000	1	160,000
Component 32	110,000	1	110,000
Component 33	115,000	2	230,000
Component 34	90,000	2	180,000
Component 35	120,000	2	240,000
Component 36	85,000	3	255,000
Component 37	100,000	1	100,000
Component 41	140,000	1	100,000
Component 42	100,000	4	400,000
Component 43	80,000	4	320,000
Component 44	150,000	4	600,000
Component 51	140,000	2	280,000
Component 52	90,000	2	180,000
Component 53	100,000	3	300,000
Component 54	130,000	1	130,000
Component 55	110,000	4	440,000
		Total Cost:	$10,278,000

The priority scheduling rule used throughout this paper here is the NTINV2-DBL rule from [4]. This scheduling rule gives the highest priority to the component with the lowest value of net inventory, which includes the number of components of the same type further along in the repair process. This rule outperformed all other rules examined in [4]. Note that under this (or a similar) rule, repair priorities and therefore repair times are directly affected by the number of component spares of each type stocked.

4. HEURISTICS FOR IMPROVED SPARES VECTORS

In this section we begin with the initial spares vector described above and attempt to obtain an approximate "efficient frontier" trading off reduced investment in inventory against higher engine backorders. To achieve this objective, spares are removed, one at a time, from the original spares vector and each resulting spares vector is simulated.

To determine analytically the mean delay days resulting from each potential removal of a spare, it is necessary to have *independent* repair times. Since repair times in this model are *dependent*, the exact changes in mean delay days cannot be determined analytically. Therefore, it is necessary to develop a heuristic index which will be used to determine a removal sequence to be applied to the original spares vector. Six different indices were developed for this purpose (see Table 3). A detailed description of each follows.

TABLE 3 — *Definition of Heuristic Indices*

Index Symbol	Definition
I_1	$(C\,S)/\lambda$
I_2	$(C\,S)/(\lambda\,W)$
I_3	$(C\,S)/(\lambda\,\mathrm{OPS}) = I_1/\mathrm{OPS}$
I_4	$S/(\lambda\,W)$
I_5	$(C^2\,S)/(\lambda\,W)$
I_6	$C\,[S/(\lambda\,W)]^2$

where:

C = unit cost of the component or module
S = number of spares of component or module stocked
λ = daily failure rate of component or module
W = total work content of component or module (processing time and estimated queue time)
OPS = number of operations required to return the component to a serviceable state.

Index one (I_1) is the ratio of the unit component cost times the current number of spares divided by the daily failure rate. Thus the index equals current investment in component spares inventory divided by the failure rate, or $-days of investment per failure. The intuitive motivation for this index is that both unit cost and the current spares level should be directly related to a willingness to delete a spare unit, while the failure rate should be inversely related. The specific form of the index argues (heuristically, with no optimization claims) that $-days of inventory investment per failure should tend to be equalized across items.

The index is calculated for each of the modules and components and then ranked from highest to lowest (as are all subsequent indices). One spare of the item with the largest index value is removed from the current spares vector. As stated above, this results in the removal of a unit having either a relatively high unit cost, a relatively high current number of spares, or a relatively low failure rate.

Index two (I_2) is I_1 divided by the total work content (total expected shop processing time, including estimated queue time) for each component type. Here total work content is used as a surrogate for total repair time. Application of this index to modules and engines is ambiguous since total work content for them is not well-defined. The actual work required for each of these is merely assembly, but there may be a delay waiting for a lower-indenture-level part which increases the total expected processing time. Since there is only a single component failure in each module, the module work content is estimated by adding the actual module assembly time to a weighted average of the total expected processing time for each of the

subordinate components.* A similar procedure is used to calculate the average work content for an engine. Thus, at all three levels, total work content is a surrogate measure for actual repair time. Since the actual repair time for two identical units may be quite different with priority scheduling, the work content measure may not be accurate.

The I_2 index also lends itself to an intuitive interpretation. Since λ is the expected daily failure rate, S/λ equals the expected days of coverage of demand that S spares provide for a given unit. Division by W expresses this coverage in units of work content. Thus, $S/\lambda \times W$ is a dimensionless value which weights the cost of each item by its approximate demand coverage given the current spares vector. If two items have equal coverage (in units of work content), the more expensive item is removed first from the spares vector.

Index three (I_3) follows the same logic as I_2, but uses the number of operations required as the surrogate for repair times. This index assumes a monotonic relationship between the number of operations and the total repair time required. (In reality, there may be a positive correlation between C, the cost of the item, and the number of operations required. In this model, these values have been generated independently.) As the number of operations increases, the amount of queuing also tends to increase. Therefore, total expected repair time can also be expected to increase since it is the sum of queuing time and processing time.

The fourth index (I_4) is similar to I_2, but it ignores the cost of the unit. This index removes items with relatively low failure rates or total work content, regardless of their cost.

Index five (I_5) and Index six (I_6) are extensions of I_2. Index five gives more weight to unit cost by squaring it.† This results in the removal of more higher-cost items earlier in the removal process than if I_2 were used. For a given budget level, the total number of spares stocked over all items tends to be slightly higher, compared to I_2.

Index six assigns an exponent of 2 to the coverage value: $(S/\lambda \times W)^2$. This is a method for examining the effect of removing more of the items with better coverage earlier in the removal process.‡

Results

Each index was used to remove spares, one at a time, from the original spares vector. The resulting spares vector was then simulated. The resulting mean delay day values for each of the indices are shown in Figure 2 for various budget levels. (Each observation point represents the outcome of one of these simulations.) Index 5, similar to Index 2 but with the unit cost squared, provides a different spares removal pattern, but produces results similar to those of Index 2. Index 1, which does not consider work content, and Index 3, where the number of operations is used as a measure of work content, perform almost 0.50 days worse, on average, when the budget is decreased by $1 million. Index 4 did not show any promise for improvement in the first seven iterations and was eliminated from further testing. The results

*Since a needed component will sometimes be available from component inventory, this measure of module (and engine) work content will generally overestimate actual processing time.

†Powers greater than 2 were examined, but with the high cost values in this problem, the C^x term dominates and the index degenerates into one using only unit cost as the removal criterion.

‡Powers of 2 and 4 were examined, but the resulting spares vectors have considerable overlap. Only results concerning the power of 2 are presented here.

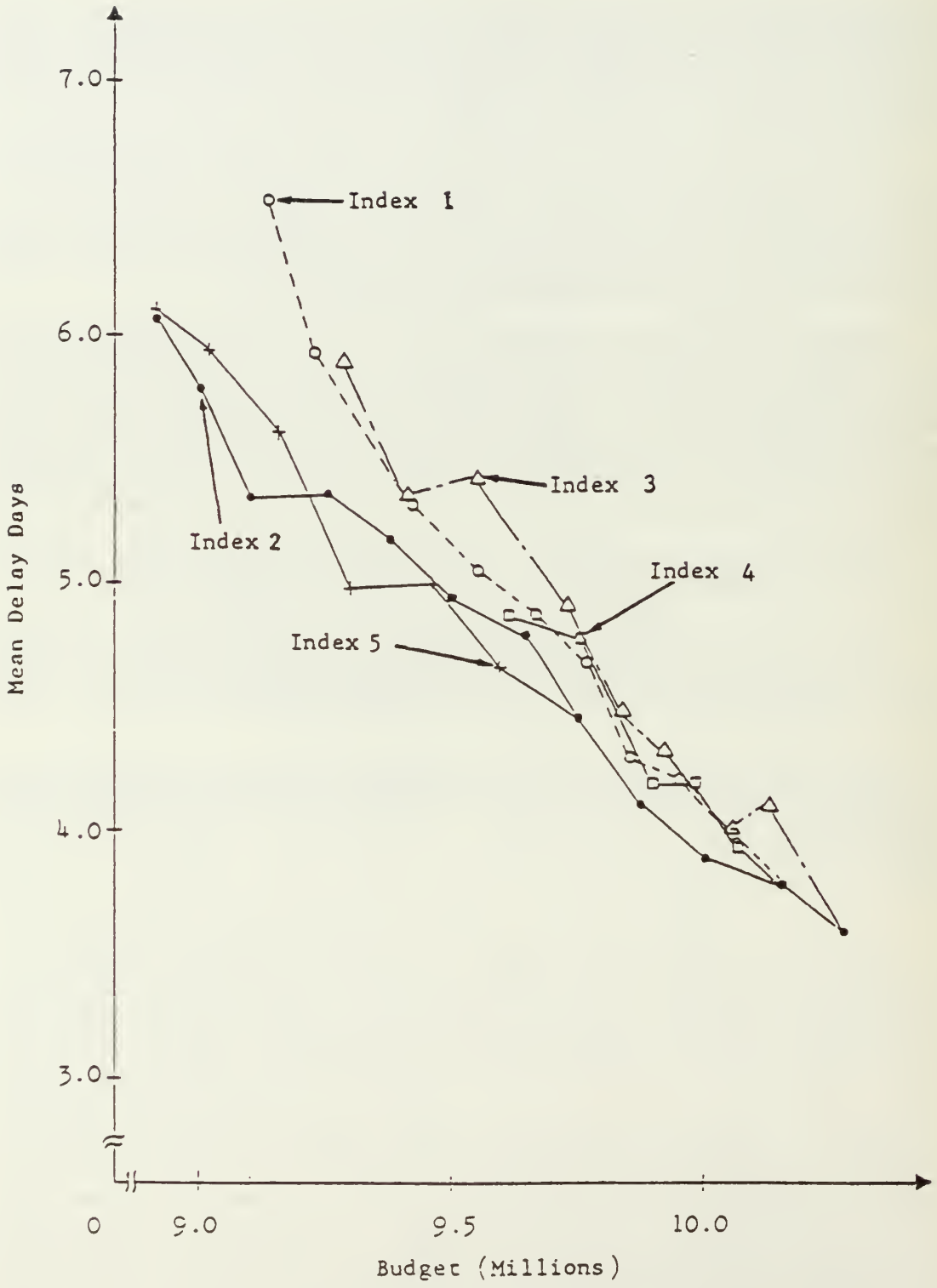

FIGURE 2. Mean delay days for five indices for various budget levels.

for Index 6 are not shown in Figure 2 due to their similarity to other results; at \$9.6 million the results coincide with those of Index 2, and below that budget level, Index 6 coincides with Index 4 in several cases.

Based on Figure 2's results, Index 2 and Index 5 seem to be the best indices of those examined here for this problem.

5. IMPROVED PERFORMANCE WITH THE SAME BUDGET

Since the theoretical model used to determine the initial spares vector requires independent repair times, it seems likely that improved delay-day performance could be possible for the same budget of \$10.3 million. Two different approaches are used to explore this issue.

Spares Convergence Using Muckstadt's Approximation

One approach for testing whether improved delay-day performance can be obtained is to use Muckstadt's Approximation in a manner for which it was not designed, namely, iteratively with the mean times obtained using a dynamic scheduling rule. (Note that the initial spares vector is based on the mean repair times obtained using the FCFS scheduling rule.) Thus the original spares vector of Table 2 was resimulated using the NTINV2-DBL priority scheduling rule. The resulting (different) set of mean repair times was then used to calculate a new spares vector using Muckstadt's Approximation with enumeration as described earlier. This new spares vector was then simulated (using the NTINV2-DBL rule) and the entire process repeated until the same spares vector was obtained in two successive iterations. Although the mean repair time for any component will vary on successive runs since the scheduling rule being used is dependent on the number of spares stocked, at some iteration the variations may not be sufficient to alter the spares vector. Convergence is defined to occur when two successive spares vectors are identical.

In Table 4, the results of applying this procedure are shown. S_0 is the initial spares vector shown in Table 2. S_4 is the final spares vector that is obtained when either the $\bar{R}_4$ or the $\bar{R}_5$ repair times are used as input to the Muckstadt procedure.*

There are two major observations to be made from Table 4. First, the mean delay-day value to which this procedure converges (3.88) is actually worse than the original value (3.80). While this is unlikely to be a statistically significant difference, it does show that the use of the FCFS mean repair times in calculating the original spares vector provides relatively good performance.

The second observation relates to the apparent paradox which exists when attempting to use any procedure based on independent repair times when repair times are actually dependent and are at least partially based on perceived need. The NTINV2-DBL priority scheduling rule is based directly on spare inventory levels. As the number of spares of a component increases, the component receives relatively lower priorities in the repair shop. These in turn lead to longer observed repair times (i.e., with more spares, net inventory is higher and module needs

*$\bar{R}$ values for engines and modules are calculated in the model and thus are not required as input data for this convergence testing.

TABLE 4 — *Convergence of Spares Vectors Using Muckstadt's Approximation Iteratively*

Iteration	0		1		2		3		4		5
Item	S_0	$\bar{R}_1$	S_1	$\bar{R}_2$	S_2	$\bar{R}_3$	S_3	$\bar{R}_4$	S_4	$\bar{R}_5$	S_5
Engine	0	—	0	—	0	—	0	—	0	—	0
Module 1	2	—	1	—	1	—	1	—	1	—	1
Module 2	1	—	1	—	2	—	1	—	1	—	1
Module 3	2	—	2	—	2	—	2	—	2	—	2
Module 4	2	—	2	—	2	—	2	—	2	—	2
Module 5	2	—	1	—	1	—	1	—	1	—	1
Component 11	2	32.9	2	27.4	2	28.4	2	27.5	2	27.3	2
Component 12	3	112.8	4	119.7	4	117.3	4	113.7	4	111.5	4
Component 13	2	37.2	2	33.7	2	32.9	2	33.0	2	33.5	2
Component 14	3	111.1	4	117.2	4	113.8	4	111.9	4	111.7	4
Component 21	2	38.8	2	32.4	1	23.6	1	23.2	1	23.8	1
Component 22	4	121.5	5	133.0	6	159.4	7	187.2	8	214.0	8
Component 23	0	21.8	0	22.1	0	21.6	0	21.8	0	22.8	0
Component 31	1	29.4	1	24.5	1	26.2	1	24.2	1	24.4	1
Component 32	1	41.7	1	33.7	0	33.4	0	30.6	0	32.6	0
Component 33	2	44.9	2	39.3	2	37.2	2	36.7	2	36.2	2
Component 34	2	36.8	3	38.2	2	27.6	2	26.6	2	25.9	2
Component 35	2	45.8	3	50.8	3	49.9	3	48.5	3	47.4	3
Component 36	3	49.5	3	41.6	3	40.0	3	39.9	3	38.0	3
Component 37	1	24.7	1	22.3	1	22.6	1	21.5	1	21.6	1
Component 41	1	21.9	1	18.5	1	17.6	1	16.3	1	16.9	1
Component 42	4	57.7	5	64.7	5	64.0	5	62.3	5	61.2	5
Component 43	4	104.3	5	109.9	5	110.3	6	130.3	6	128.0	6
Component 44	4	48.4	5	56.3	5	55.4	5	54.6	5	53.8	5
Component 51	2	25.0	2	19.9	2	20.3	2	19.0	2	19.0	2
Component 52	2	39.0	2	33.8	2	31.6	2	30.3	2	29.8	2
Component 53	3	63.7	4	69.9	4	71.0	4	69.4	4	68.3	4
Component 54	1	29.7	1	27.1	1	28.0	1	27.3	1	27.3	1
Component 55	4	129.1	5	140.5	5	142.4	5	135.2	5	134.0	5
Mean Delay Days	3.80		3.88		3.85		3.91		3.88		3.88

for components are satisfied more frequently from stock). A procedure like Muckstadt's which relies on the independence of these repair times will purchase additional spares of an item if the repair time increases significantly. Therefore, a feedback loop is created in which more spares are purchased, lower priorities and longer repair times result, and possibly more spares are purchased.

In Table 4 this behavior is observed for component 22. The number of spares for this component is increased by one in each iteration and the mean repair time increases each time as well. Since the marginal decrease in expected backorders is diminishing as each additional spare is added, the procedure eventually converges to the S_4 spares vector.*

Index Equalization

A second method which was examined to determine a better spares vector for a given budget is *Index Equalization.* Index Equalization is a process where the number of spares of each component type are set such that the index number values (as defined in Table 3) for all components are approximately equal. For example, if Index 2 $(C\,S)/(\lambda\,W)$ was being used, $C/(\lambda\,W)$ would be calculated for each component. (This value does not vary with the number of spares). Then the number of spares, S, is increased for each component until Index 2 value is as close as possible to some predetermined value. (Due to the integrality of the spares stocking levels allowed, it is not possible for all index values to be precisely equal.)

Since Index 2 and Index 5 provided similar performance in spares reduction, they were selected for initial Index Equalization testing. Each index was used to calculate component spares for budgets of \$10.3 million (the original budget), \$9 million and \$8 million. A particular budget level is obtained by varying the predetermined target index equalization value until enough spares are stocked to match the desired budget. For example, if an index value of 500 results in a budget of \$7 million being used, increasing the index value to 600 will increase the budget used.

The resulting spares vectors were simulated using the NTINV2-DBL rule with the resulting mean delay-day values shown in Figure 3. (For these tests, the engine and module spares remained at the levels shown in Table 2.) Index 2 performed slightly better than Index 5 at budgets of \$8 million and \$10.3 million, while at \$9 million they were essentially equal. Although both index equalization procedures provided marginally improved performance over the original spares vector, it should be noted that Muckstadt's Approximation performs almost as well, even though it assumes the dependent repair times from the FCFS scheduling rule are independent. Index 2 was selected for further testing and is used for all remaining results in this section.

When using Index Equalization there are two issues of importance related to the accuracy of the parameters being used in calculating the indices. These are the estimation of repair times by work content and the lack of a good measure of work content for modules and engines. The first issue was discussed when the indices were defined. The work content for modules and engines clearly must include the actual assembly times (2.5 days for engines, 5.0 days for all modules). However, these alone are not adequate to reflect relative repair times, since repair times of modules (or engines) may also consist of waiting time while the proper components (or modules) are repaired.

As stated above, one possible procedure is to estimate module work content as the 5-day assembly time plus a weighted average of the work content of the components comprising the module. For example, if a module consists of only 2 components with relative failure rates of

*The convergence procedure was tested for additional scheduling rules with similar results. Convergence in all cases was achieved in 5 or fewer iterations. In other situations convergence may not occur; cycling may take place. The convergence question is moot since the performance of this approach is actually degraded.

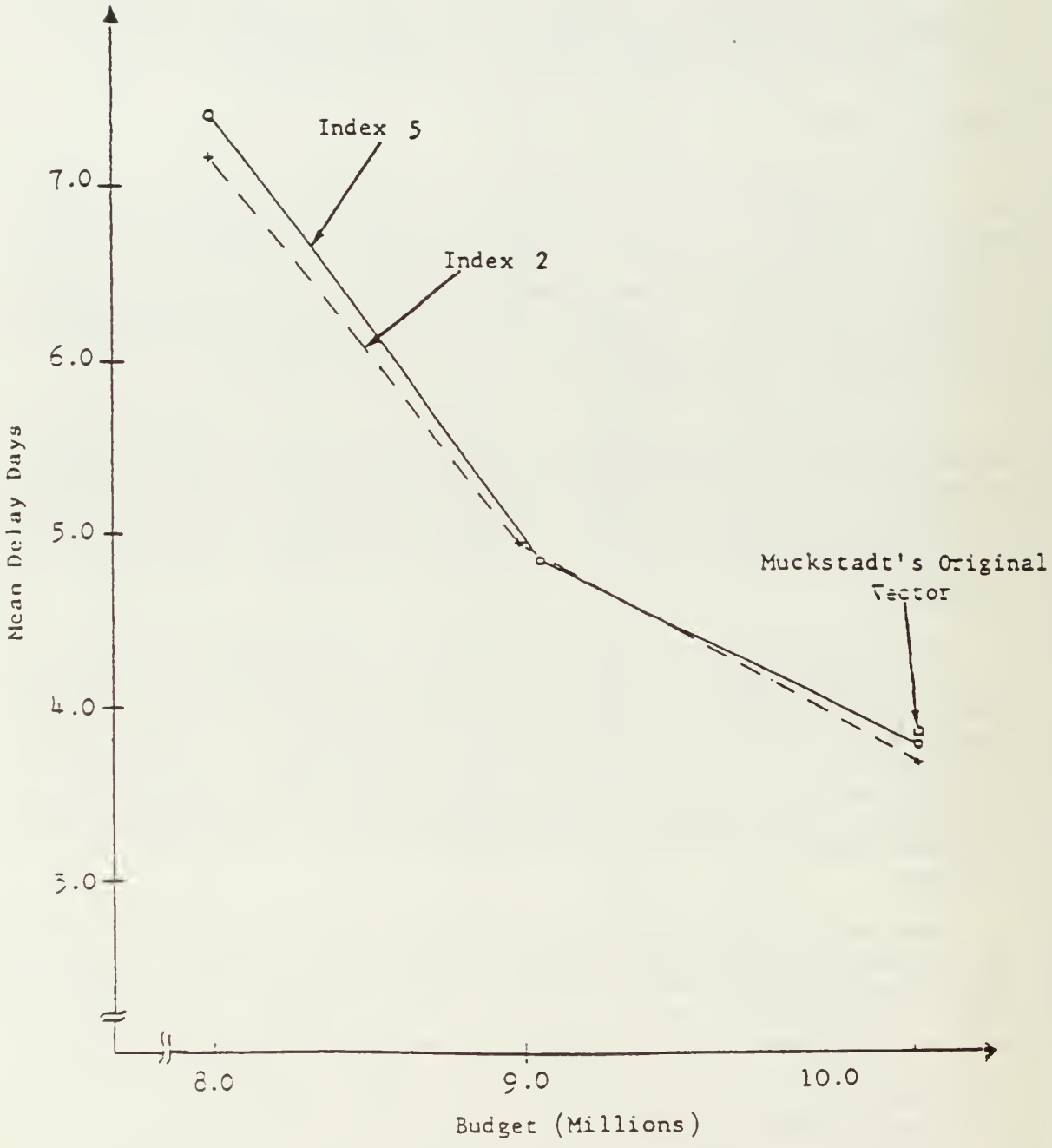

FIGURE 3. Index Equalization Performance

0.25 and 0.75 and work content of 2 days and 4 days, respectively, the module work content would be estimated as $5 + (0.25)(2) + (0.75)(4) = 8.5$ days. This value overstates the module work content at least 25% of the time (i.e., when the first component fails). Also, when there is a positive on-hand inventory of the required component, the actual repair time is only 5 days. If the system does stock spares of the component, the module demand may also be filled by a component which has already completed one or more operations. Clearly, actual module repair times are a function of the number of component spares and a surrogate measure, such as work content, is not easily calculated. For these reasons, another approach will be examined below.

Since Muckstadt's Approximation appeared to provide a good spares vector for the \$10.3 million budget, it might also be reasonably good at other budget levels. Thus the approximation was used to calculate the spares vectors for budgets of \$8 millions, \$8.5 million, \$9 million, and \$9.5 million using the FCFS mean repair times. Index 2 equalization was also used to calculate spares vectors for these four budget levels, but here the Index Equalization technique was restricted to have constant module spares, while Muckstadt's method applied to modules as well as components decreases total module spares by one for each budget decrease. These spares vectors were simulated and the resulting mean delay days are plotted in Figure 4. With fixed module spares, Index Equalization performance degrades relative to Muckstadt's Approximation as the budget is decreased. This is due to overspending on modules at the lower budget levels.

As an alternative, Index Equalization was used for components only, while the module spares were those generated by Muckstadt's Approximation. When simulated, these spares vectors resulted in the delay-day values plotted in Figure 5. In this case, Index Equalization provides better performance in all cases. Therefore, it appears that a good procedure is to use Muckstadt's model with enumeration to determine the upper level spares (engines and modules) and Index Equalization to allocate the remaining budget to the purchase of spare components.*

6. SENSITIVITY ANALYSIS OF BUDGET ALLOCATION TO ENGINE, MODULE, AND COMPONENT SPARES LEVELS

When repair times are independent, an analytical model like that of Muckstadt will determine an optimal allocation of the available budget between engine, module and component spares, as well as optimal allocations of spares within each level. It is instructive to explore the sensitivity of spares vector performance to deviations from the optimal allocation of the budget among the three levels (whether repair times are independent or dependent) while allocating appropriately within each level. This sensitivity was explored for the dependent repair times which exist in this model.

In this portion of the study, engine and module spares were varied one at a time and component spares for the remaining budget were calculated using Index 2 equalization as previously described. A second index number was used to calculate the desired module vector for each

*Since module and engine spare levels are based on an expected resupply time which consists of a constant assembly time plus the lower level expected delay, Muckstadt's method (which assumes independent repair times) should provide reasonable performance when the assembly time is a major percentage of the overall expected resupply time.

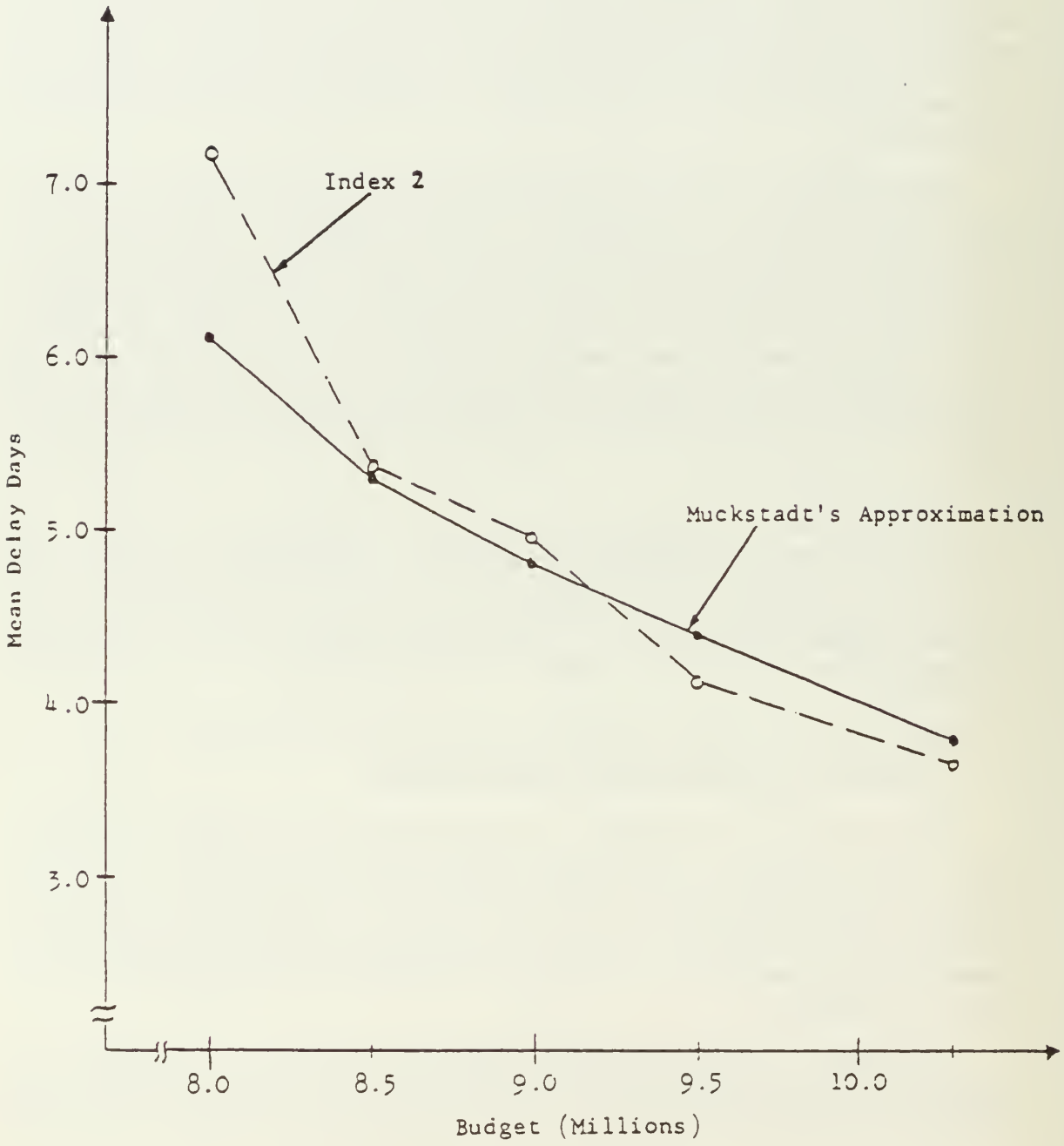

FIGURE 4. Comparison of Muckstadt's Approximation and Index 2 equalization with fixed module spares.

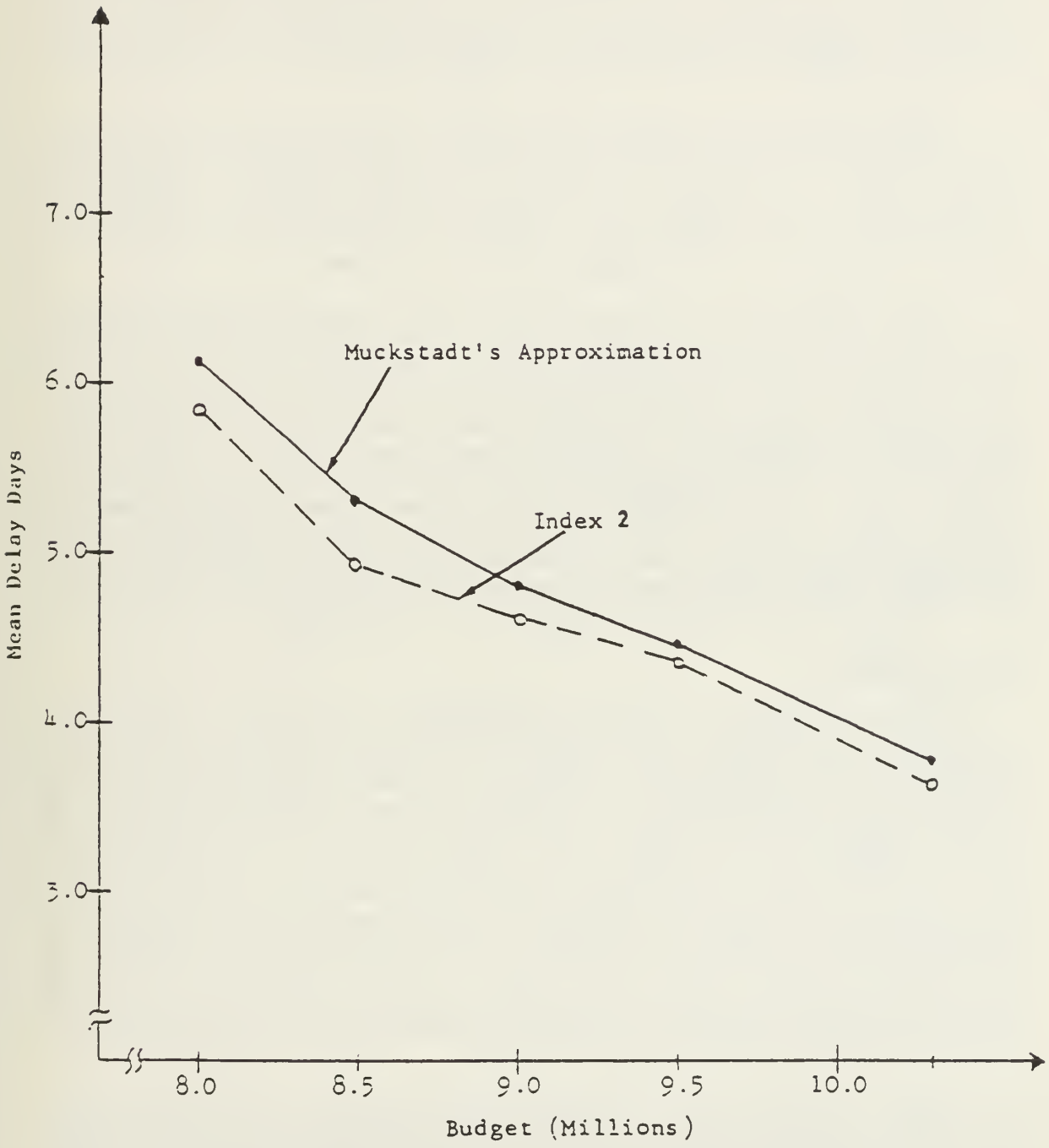

FIGURE 5. Comparison of Muckstadt's Approximation and Index 2 equalization with variable module spares.

iteration. This index is based on the ratio of the cost (C) and overall failure rate (λ) for each module. (These values are given in Table 2 and Table 1 respectively.) Specifically, this ratio was used to determine which module spare should be the next to add when varying the module spares vector upward from zero. For example, if $C_1/\lambda_1 = 3$ and $C_2/\lambda_2 = 4$, module 1 would be added before module 2. Engine spares were initially zero (the value used in all previous simulations); engine spare levels of one and two were also examined.

Several spares vectors were generated for each of the engine spares levels. Each vector was simulated using the NTINV2-DBL scheduling rule and the resulting mean delay days are plotted in Figure 6. Each point in the upper half of Figure 6 represents the observed mean delay days for 0, 1, or 2 engines, with the remaining budget allocated to components and modules as shown in the lower quadrant. (The lines in the lower quadrant are budget lines; addition of 1 engine shifts the line towards the origin by $2 million.)

The best performance (3.61 delay days) occurs when one engine is stocked and approximately $5.4 million is spent on component spares. However, this is only slightly better than two of the spares vectors with no engines (3.62 days and 3.63 days) and the next spares vector with one engine (3.65 days) which spends $5.9 million on components. The original Muckstadt module spares, with Index 2 equalized component spares, performs almost as well (3.67 days). Thus, there are five spares vectors, each with component spares budgets in a range of $5 million to $6 million, which result in nearly identical performance. As more money is allocated to component spares, the performance with zero engine spares degraded much more slowly than when one engine is stocked. When less than $5 million is spent on components, these two engine stocking levels provide similar results over the range examined. Stocking two engine spares clearly degrades performance, with minimum mean delay days of 5.25, more than 1.5 days worse than the minimum with one engine.

CONCLUSIONS

In this article we have examined different methods of determining a desirable spares inventory choice when repair times are dependent. Muckstadt's Approximation was used iteratively with the dependent repair times obtained using a dynamic priority scheduling rule until convergence to a single spares vector was obtained. However, no improved spares vector was generated at any of the five iterations. Index Equalization was applied to the component spares vector and performance was improved by 0.13 days, on average — only a marginal improvement. Using Muckstadt's Approximation, spares vectors were generated for various budget levels. The engine and module spares levels generated were used, but Index Equalization was used to determine the component spares levels. The equalized spares vectors provided somewhat better performance than Muckstadt alone in all cases. However, it was noted that Muckstadt's model appears to be very robust, even with dependent repair times. The robustness of the METRIC model when expediting was used has been shown by Sherbrooke [12]. The results presented here also support this conclusion.

Finally, it was shown that several equally good spares vectors exist for this problem. While engine and module spares for these better vectors vary widely, the component budgets lie in a narrower range (5 to 6 million dollars). Thus, the main tradeoff flexibility for the better spares vectors occurs at the engine and module levels when appropriate within-level allocations are made.

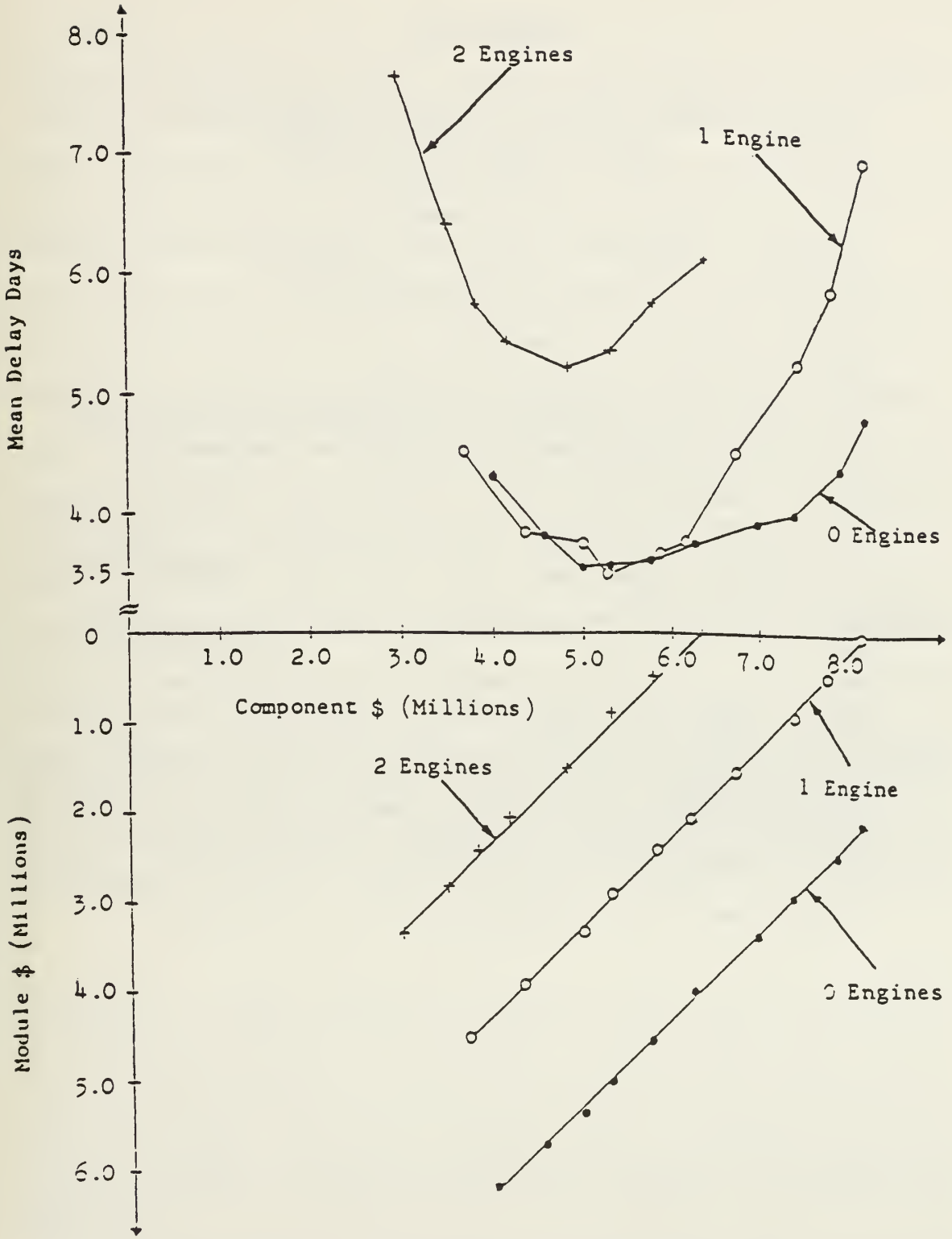

FIGURE 6. Sensitivity of mean delay day performance to budget allocation decisions.

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RELIABILITY APPLICATIONS OF THE RELEVATION TRANSFORM

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ABSTRACT

It is shown that Krakowski's relevation transform generates the nonhomogeneous Poisson process in an analogous fashion to the way in which Stieltjes convolution generates the renewal process. Properties of failure rates and of inter-failure times are discussed and an application to warranty analysis is described.

1. INTRODUCTION

The (Stieltjes) convolution of two distribution functions with support on the nonnegative axis, F and G say, is denoted

$$F * G(t) = \int_0^t F(t-u)\, g(u)\, du$$

if $g = G'$ exists. It is the distribution function of the time to failure of the second of two components when the second component (with life distribution G) is placed in service on the failure of the first (with life distribution F); the replacement component is assumed to be new on installation. Suppose, however, that we replace the failed component by one of equal age. The survivor function of the time to system failure (i.e., both components are failed) is the *relevation* of $\bar{F}$ and $\bar{G}$, the survivor functions of the first and second components, respectively, and is denoted

$$\bar{F} \# \bar{G}(t) = \bar{F}(t) + \int_0^t \frac{\bar{G}(t)}{\bar{G}(u)} f(u)\, du$$

assuming that $f = F'$ exists. (The assumption that f and g exist is not necessary, but is typically reasonable in practice and simplifies the presentation.) This concept was introduced by Krakowski [11] who studied some properties of what he calls the *relevation transform.* In this paper we consider the stochastic process generated by the successive failures of a component which, on failure, is replaced by a component of equal age or, equivalently, in which the component is instantaneously restored to its condition immediately prior to failure. The latter interpretation provides us with a model of the failure pattern of a component for which the repair time is negligible in comparison with the failure time, which may be more realistic than the renewal process in certain instances. A similar model is described by Ascher [1]; see also Thompson [14].

Further developments of the relevation transform are discussed by Johnson and Kotz [9]; [10].

2. THE TIME TO THE nth FAILURE

Let $\bar{F}_{(n)}$ denote the survivor function of X_n, the time to the nth failure. Clearly,

$$\bar{F}_{(n)}(t) = \begin{cases} 1 - U(t) & n = 0 \\ \bar{F}(t) & n = 1, \\ \bar{F}_{(n-1)} \# \bar{F}(t) & n \geqslant 2 \end{cases}$$

where F is the life distribution of a new component and U is the Heaviside function. We can interpret $\bar{F}_{(n)}(t)$ as the survivor function of

(a) the time to the nth failure given that a failed component is replaced by one of equal age

(b) the time to the nth failure given that instantaneous repair to the condition immediately prior to failure is performed at each breakdown

(c) the time to the nth failure of a system for which there is an arbitrarily large number of spares on hot standby.

An expression for $\bar{F}_{(n)}(t)$ follows on observing that

$$\bar{F}_{(n)}(t) = \sum_{k=0}^{n-1} p\{k \text{ failures in } (0, t]\}$$

and hence

$$\bar{F}_{(n)}(t) = \bar{F}(t) + \sum_{k=1}^{n-1} \int_0^t \frac{f(u_1)}{\bar{F}(u_1)} \int_{u_1}^t \frac{f(u_2)}{\bar{F}(u_2)} \cdots \int_{u_{k-1}}^t \frac{f(u_k)}{\bar{F}(u_k)} \bar{F}(t) \times du_k \ldots du_2 du_1 \tag{1}$$

for $0 = u_0 < u_1 < u_2 < \ldots < u_k < t$. Define $\lambda(x) = f(x)/\bar{F}(x)$ (the hazard function) where $f = F'$ and

$$\Lambda(x) = \int_0^x \lambda(u)\, du = -\log \bar{F}(x)$$

(the cumulative hazard function). After some straightforward but lengthy algebra, (1) simplifies to

$$\bar{F}_{(n)}(t) = \bar{F}(t) \sum_{k=0}^{n-1} [\Lambda(t)]^k / k! \tag{2}$$

Krakowski [11] proves (2) by means of an induction argument.

If $\nu(t)$ denotes the number of failures in $(0, t]$,

$$p\{\nu(t) = n\} = \frac{\bar{F}(t)[\Lambda(t)]^n}{n!} \quad (n = 0, 1, 2, \ldots),$$

i.e., $\{\nu(t), t > 0\}$ forms a nonhomogeneous Poisson process with intensity function $\lambda(t)$. (This result also follows by making an obvious transformation of the time scale of a Poisson process and is noted by Barlow, Proschan and Scheuer [3]; see also Parzen [13], page 126.)

Thus, we see that the relevation transform generates the nonhomogeneous Poisson process in a manner analogous to the way in which the convolution operation generates the

renewal process. Observe that the sequence of failures generated by the relevation transform constitutes a renewal process if and only if F is exponential; this follows immediately from the characterization theorem of Grosswald, Kotz and Johnson [8].

On differentiating (2), we obtain the following expression for $f_{(n)}(t)$, the density of X_n:

$$f_{(n)}(t) = \frac{f(t)[\Lambda(t)]^{n-1}}{(n-1)!} \qquad (n = 1, 2, 3, \ldots), \tag{3}$$

defining $f_{(0)}(t)$ to be the Dirac delta function. Substituting

$$\Lambda(t) = \sum_{k=1}^{\infty} [F(t)]^k/k,$$

which follows from the identity $-\log(1-x) = \sum_{k=1}^{\infty} x^k/k$, into (3), we see that

$$f_{(n)}(t) = \sum_{k=n}^{\infty} a_k\, k f(t)\,[F(t)]^{k-1}$$

for suitable constants $\{a_k\}$, i.e., $f_{(n)}(t)$ is an infinite mixture of the densities of the largest order statistics of a random sample size $k (k = n, n+1, n+2, \ldots)$ from the population with distribution function F.

On integrating (3), we obtain the well-known identity

$$E\{[\Lambda(X)]^n\} = \Gamma(n+1),$$

where X is the random variable with distribution function F; it follows from this and (2) that

$$E\{\bar{F}_{(n)}(X)/\bar{F}(X)\} = n.$$

The joint distribution of X_n and X_{n+m} is readily found. Clearly, the survivor function of X_{n+m} conditional on $\{X_n = t\}$ is

$$\bar{F}_{(m)}(\tau \mid t) = \frac{\bar{F}(\tau)}{\bar{F}(t)} \sum_{k=0}^{m-1} \{\Lambda(\tau) - \Lambda(t)\}^k/k!$$

and hence the joint density of X_n and X_{n+m} is

$$f_{(n,m)}(t,\tau) = \frac{1}{(n-1)!\,(m-1)!} f(\tau)\lambda(t)[\Lambda(t)]^{n-1}[\Lambda(\tau) - \Lambda(t)]^{m-1} \tag{4}$$

$$\text{for } 0 < t < \tau < \infty.$$

EXAMPLES:

(i) Suppose F is Weibull with unit scale parameter, i.e., $F(t) = 1 - \exp(-t^\alpha)$. Then

$$f_{(n)}(t) = \frac{\alpha t^{n\alpha-1} e^{-t^\alpha}}{(n-1)!},$$

i.e., a member of the generalized gamma family.

(ii) Suppose F is Lomax with unit scale parameter, i.e., $F(t) = 1 - 1/(1+t)^\alpha$. Then

$$f_{(n)}(t) = \frac{\alpha^n[\log(1+t)]^{n-1}}{(1+t)^{\alpha+1}(n-1)!}.$$

A general formula for the rth moment of X_n is not available, but expressions can be found for certain special cases. Thus, for example, for the Weibull distribution with unit scale parameter,

$$E(X_n^r) = \Gamma(n + r/\alpha)/\Gamma(n).$$

Note that, as would be expected, $E(X_1^r) = \Gamma(1 + r/\alpha)$, the rth moment of the Weibull distribution.

3. MONOTONE FAILURE RATES

A probability distribution function is said to be IFR (DFR) if the corresponding hazard function is nondecreasing (nonincreasing). In this section we consider the effects of the relevation transform on monotone failure rates.

Let

$$\lambda_{(n)}(t) = f_{(n)}(t)/\bar{F}_{(n)}(t)$$

denote the hazard function of the distribution $F_{(n)}(t)$. Clearly,

$$\lambda_{(2)}(t) = \frac{\lambda(t)\Lambda(t)}{1+\Lambda(t)}$$

and hence $\lambda_{(2)}(t) < \lambda(t)$ for all t, irrespective of the functional form of F, as would be expected from the definition of the relevation transform. More generally, it is easily seen that, for $n = 1, 2, 3 \ldots$, $\lambda_{(n+1)}(t) < \lambda_{(n)}(t)$. (We assume that $F(t) > 0 \; \forall \; t > 0$.)

We now show that the n-fold recursive relevation of F is IFR whenever F is IFR (an analogous result is given by Krakowski [11]).

THEOREM 1: Suppose F is IFR. Then $F_{(n)}$ is IFR for $n = 1, 2, 3, \ldots$.

PROOF: The proof is by induction on n; if $F_{(n)}$ is IFR, it is not difficult to show that $\lambda_{(n+1)}(t+\delta) - \lambda_{(n+1)}(t) \geqslant 0$ for all $t > 0$ and $\delta > 0$. The algebra is tedious but straightforward and is hence omitted. ||

A similar result does not hold for DFR distributions; a counterexample is provided by the Weibull distribution $F(t) = 1 - \exp(-t^{3/4})$. In this case,

$$\lambda_{(2)}(t) = \frac{3t^{1/2}}{4(1+t^{3/4})}$$

which is clearly not monotone nonincreasing.

For convolutions, the same is true, i.e., $F^{(n)}(t)$, the n-fold recursive convolution of F, is IFR if F is IFR, but DFR is not preserved under convolution. More generally, $F * G$ is IFR if both F and G are IFR ([2], page 100) so it seems plausible by analogy that $1 - \bar{F} \# \bar{G}$ is also IFR. This is not, in fact, the case; a further restriction is required, motivating the following definition.

DEFINITION: For distributions F and G, the *conditional hazard function of F given G* is defined as $f(t)/\bar{G}(t)$.

THEOREM 2: If G is IFR and the conditional hazard function of F given G is nondecreasing, then $1 - \bar{F} \# \bar{G}$ is IFR. Similarly, if F is IFR and the conditional hazard function of G given F is nondecreasing, then $1 - \bar{G} \# \bar{F}$ is IFR.

PROOF: If $l(t) = -\frac{d}{dt} \log \bar{F} \# \bar{G}(t)$, it is easily seen that $l(t + \delta) - l(t) \geqslant 0$ for all $t > 0, \delta > 0$. ||

Note that we do not require $F(t)$ to be IFR in order that $1 - \bar{F} \# \bar{G}(t)$ be IFR.

4. MARGINAL DISTRIBUTIONS

Let $Y_n = X_n - X_{n-1}$ $(n = 1, 2, 3, \ldots)$ be the time from the $(n-1)$th to the nth failures, defining $X_0 = 0$. Let $B_n(t)$ and $\beta_n(t)$ denote the distribution and density functions of Y_n. Clearly,

$$\bar{B}_n(t) = \int_0^\infty \frac{\bar{F}(t+u)}{\bar{F}(u)} f_{(n-1)}(u)\, du \tag{5}$$

which, on integration by parts, yields the following expression:

$$\bar{B}_n(t) = \int_0^\infty f(t+u) \frac{[\Lambda(u)]^{n-1}}{(n-1)!}\, du. \tag{6}$$

(A similar expression is given by Parzen [13], page 128.) This is the survivor function of the marginal distribution of Y_n. The conditional distribution of Y_n given X_{n-1} has survivor function

$$\bar{F}(X_{n-1} + t)/\bar{F}(X_{n-1})$$

as noted by Barlow, Proschan and Scheuer [3] and Crow [7].

An expression for $\beta_n(t)$ follows on differentiating (6):

$$\beta_n(t) = \int_0^\infty f(t+u)\, \lambda(u) \frac{[\Lambda(u)]^{n-2}}{(n-2)!}\, du \tag{7}$$

for $n \geqslant 2$. On integrating (6), we see that

$$E(Y_n) = \int_0^\infty \bar{F}(u) \frac{[\Lambda(u)]^{n-1}}{(n-1)!}\, du. \tag{8}$$

If, for example, $F(t) = 1 - \exp(-t^\alpha)$, then

$$E(Y_n) = \frac{\Gamma(n - 1 + 1/\alpha)}{\alpha \Gamma(n)}.$$

Observe that

$$E(Y_{n+1}) - E(Y_n) = (1 - \alpha) \frac{\Gamma(n - 1 + 1/\alpha)}{\alpha^2 \Gamma(n+1)}, \text{ i.e.,}$$

$E(Y_{n+1}) \leqslant (\geqslant) E(Y_n)$ if F is IFR (DFR). This inequality also holds for other monotone failure rate distributions. In fact, a stronger result may be obtained.

THEOREM 3: The sequence $Y_1, Y_2, Y_3, \ldots$ is stochastically nonincreasing (nondecreasing) if F is IFR (DFR).

PROOF: Suppose F is IFR. Then $\lambda(u) \leqslant \lambda(u+t)$ for all $u > 0$ and $t > 0$ and hence

$$\begin{aligned}\bar{B}_{n+1}(t) &= \int_0^\infty \bar{F}(t+u)\lambda(u)\frac{[\Lambda(u)]^{n-1}}{(n-1)!}\,du \\ &\leqslant \int_0^\infty \bar{F}(t+u)\lambda(t+u)\frac{[\Lambda(u)]^{n-1}}{(n-1)!}\,du \text{ for all } t \\ &= \bar{B}_n(t) \text{ from (6)},\end{aligned}$$

i.e., the sequence $Y_1, Y_2, Y_3, \ldots$ is stochastically nonincreasing. Similarly, if F is DFR, the sequence is stochastically nondecreasing. ||

5. APPLICATION TO WARRANTY ANALYSIS

The nonhomogeneous Poisson process provides a (possibly) more realistic model of a repairable machine than the renewal process and, in certain instances, is mathematically more tractable. An important example is the Blischke-Scheuer free-replacement warranty model [4], [5].

Suppose that the lifetime of an item follows a distribution F with mean μ and that an item is required by a consumer for a length of time L. On failure before L, the item is replaced free if it fails during the warranty period of length W, otherwise a new item is purchased at a cost K. Let

$$M(t) = \sum_{r=1}^{\infty} F^{(r)}(t)$$

denote the renewal function corresponding to F, recalling that $F^{(r)}$ is the r-fold recursive convolution of F. Then the *average* number of times payment must be made under the free replacement warranty is

$$1 + L/\{\mu[1 + M(W)]\}.$$

Blischke and Scheuer [4] show that the breakeven cost at which the customer would be indifferent to buying the item with or without a warranty is

$$C_1^* = \frac{\mu K[1 + M(L)][1 + M(W)]}{\mu[1 + M(W)] + L}.$$

For the nonhomogeneous Poisson process, the corresponding expression is

$$\hat{C}_1^* = \frac{K[1 + \Lambda(L)][W\bar{F}(W) + D_W]}{(W + L)\bar{F}(W) + D_W} \tag{9}$$

where $D_W = \int_W^\infty \bar{F}(u)\,du$. A sharper result can, however, be obtained in this case; each failure immediately succeeding the elapse of the warranty is a regeneration point in an embedded renewal process with inter-renewal times governed by the distribution function

$$A_W(t) = \frac{F(t + W) - F(W)}{1 - F(W)}. \tag{10}$$

Thus, the *expected* number of payments under the free-replacement warranty is

$$1 + \sum_{r=1}^{\infty} A_W^{(r)}(L)$$

which may readily be evaluated by means of the Cléroux-McConalogue cubic splining algorithm [6], [12]. The breakeven cost at which the consumer would be indifferent to purchasing with or without a warranty is

$$\hat{C}^* = \frac{K[1 + \Lambda(L)]}{1 + \sum_{r=1}^{\infty} A_W^{(r)}(L)}. \tag{11}$$

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THE HNBUE AND HNWUE CLASSES OF LIFE DISTRIBUTIONS

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ABSTRACT

Different properties of the HNBUE (HNWUE) class of life distributions (i.e., for which $\int_t^\infty \bar{F}(x)\,dx \leqslant (\geqslant)\, \mu \exp(-t/\mu)$ for $t \geqslant 0$, where $\mu = \int_0^\infty \bar{F}(x)\,dx$) are presented. For instance we characterize the HNBUE (HNWUE) property by using the Laplace transform and present some bounds on the survival function of a HNBUE (HNWUE) life distribution. We also examine whether the HNBUE (HNWUE) property is preserved under the reliability operations (i) formation of coherent structure, (ii) convolution and (iii) mixture. The class of distributions with the discrete HNBUE (discrete HNWUE) property (i.e., for which $\sum_{j=k}^{\infty} \bar{P}_j \leqslant (\geqslant)\, \mu(1 - 1/\mu)^k$ for $k = 0, 1, 2, \ldots$, where $\mu = \sum_{j=0}^{\infty} \bar{P}_j$ and $\bar{P}_j = \sum_{k=j+1}^{\infty} p_k$) is also studied.

1. INTRODUCTION

In many applications (e.g., reliability, maintenance theory, inventory theory and biometry) different forms of aging are of interest. Therefore, during the last decades several classes of life distributions based on notions of aging have been studied. The most well-known of these classes are the IFR, IFRA, NBU, NBUE and DMRL classes (with duals), the definitions of which are given in the Appendix.

In this partly expository paper we shall study a class of life distributions named HNBUE (harmonic new better than used in expectation) and its dual class HNWUE (W = worse).

In Section 2 we present the HNBUE (HNWUE) property and a corresponding property for discrete distributions named discrete HNBUE (discrete HNWUE).

In Section 3 we give relationships between the HNBUE (HNWUE) property and the Laplace transform and the discrete HNBUE (discrete HNWUE) property and the generating function.

In Section 4 we examine whether the HNBUE (HNWUE) property is preserved under the reliability operations: (i) formation of coherent structure, (ii) convolution and (iii) mixture.

In Section 5 we present some bounds on the survival function of a HNBUE (HNWUE) life distribution.

2. DEFINITIONS

The HNBUE (HNWUE) Property

DEFINITION 1: A life distribution F and its survival function $\bar{F} = 1 - F$ with finite mean $\mu = \int_0^\infty \bar{F}(x)\,dx$ are said to be harmonic new better than used in expectation (HNBUE) if

$$\int_t^\infty \bar{F}(x)\,dx \leqslant \mu \exp(-t/\mu) \text{ for } t \geqslant 0. \tag{1}$$ □

If the reversed inequality is true, F and $\bar{F}$ are said to be harmonic new worse than used in expectation (HNWUE). This gives a dual class to the HNBUE class of life distributions in the same way as the IFR, IFRA, NBU, NBUE and DMRL classes have duals (cf. Appendix).

The HNBUE and HNWUE classes were introduced by Rolski [17]. The reason for the name HNBUE is the following. Suppose for simplicity that $\bar{F}(t) > 0$ for $t \geqslant 0$ and let

$$e_F(t) = \int_t^\infty \bar{F}(x)\,dx / \bar{F}(t)$$

denote the mean residual life of a unit of age t. Then the inequality (1) can be written

$$\frac{1}{\frac{1}{t}\int_0^t \{e_F(x)\}^{-1}dx} \leqslant \mu \text{ for } t \geqslant 0. \tag{2}$$

This inequality says that the integral harmonic mean value of $e_F(x)$, $0 \leqslant x \leqslant t$, is less than or equal to the integral harmonic mean value of $e_F(0) = \mu$.

The HNBUE property can be characterized in several ways. One of these is the following.

Let T_F denote the equilibrium distribution of F defined by

$$T_F(x) = \frac{1}{\mu}\int_0^x \bar{F}(s)\,ds \text{ for } x \geqslant 0. \tag{3}$$

This life distribution is of interest e.g., in renewal theory (see e.g. Feller [8], Chapter 11). By using T_F, the HNBUE property can be written as

$$\bar{T}_F(t) \leqslant \bar{G}(t) \text{ for } t \geqslant 0,$$

where $\bar{T}_F = 1 - T_F$ and $\bar{G}(t) = \exp(-t/\mu)$, $t \geqslant 0$.

One reason why we find the HNBUE (HNWUE) class interesting is the following.

Let ξ_j, with life distribution F_j and survival function $\bar{F}_j$, for $j = 1, 2, \ldots, n$, denote the life lengths of n independent units. If μ_s and μ_p denote the mean lives of their series and parallel system, respectively, we have

$$\mu_s = \int_0^\infty \{\prod_{j=1}^n \bar{F}_j(x)\}dx \tag{4}$$

and

$$\mu_p = \int_0^\infty \{1 - \prod_{j=1}^{n} F_j(x)\}dx. \tag{5}$$

The integrals in (4) and (5) are often difficult to calculate. However, if F_j is HNBUE with mean μ_j and $\bar{G}_j(x) = \exp(-x/\mu_j)$, $x \geqslant 0$, for $j = 1, 2, \ldots, n$, we have the bounds (see Klefsjö [13])

$$\mu_s \geqslant \int_0^\infty \{\prod_{j=1}^{n} \bar{G}_j(x)\}dx \tag{6}$$

and

$$\mu_p \leqslant \int_0^\infty \{1 - \prod_{j=1}^{n} G_j(x)\}dx \tag{7}$$

which are simple to calculate.

In fact it follows from Theorem 3.2 on p. 33 in Barlow and Proschan [3] that (6) and (7) are true under the weaker condition that $\xi_1, \xi_2, \ldots, \xi_n$ are associated and each ξ_j is HNBUE with mean μ_j. That $\xi_1, \xi_2, \ldots, \xi_n$ are associated means a form of dependence (see Barlow and Proschan [3], Chapter 2).

The Discrete HNBUE (Discrete HNWUE) Property

A survival function which is IFR (DFR) is continuous except possibly at the right (left) hand end point of its interval of support (cf. Barlow and Proschan [3], p. 77). The corresponding is not true in the HNBUE (HNWUE) case. There are discrete survival functions which are HNBUE (HNWUE).

We shall now introduce another aging property which is useful when we discuss discrete distributions.

Let ξ be a strictly positive integer valued random variable and let $\bar{P}_k = P(\xi > k)$ denote the corresponding survival probabilities. Let further $1 = \bar{Q}_0 \geqslant \bar{Q}_1 \geqslant \bar{Q}_2 \geqslant \ldots$ denote the corresponding survival probabilities of a geometric distribution with finite mean

$$\mu = \sum_{j=0}^{\infty} \bar{Q}_j = \sum_{j=0}^{\infty} \bar{P}_j,$$

i.e., $\bar{Q}_k = (1 - 1/\mu)^k$, $k = 0, 1, 2, \ldots$. Since the discrete counterpart to the exponential distribution is the geometric distribution it seems natural to say that a discrete distribution with survival probabilities $\bar{P}_0 \geqslant \bar{P}_1 \geqslant \bar{P}_2 \geqslant \ldots$ is discrete HNBUE if

$$\sum_{j=k}^{\infty} \bar{P}_j \leqslant \sum_{j=k}^{\infty} \bar{Q}_j \text{ for } k = 0, 1, 2, \ldots.$$

This leads to the following definition.

DEFINITION 2: A discrete distribution and its survival probabilities $\bar{P}_k$, $k = 0, 1, 2, \ldots$, with finite mean $\mu = \sum_{k=0}^{\infty} \bar{P}_k$ are said to be discrete harmonic new better than used in expectation (discrete HNBUE) if

$$\sum_{j=k}^{\infty} \bar{P}_j \leqslant \mu \left(1 - \frac{1}{\mu}\right)^k \text{ for } k = 0, 1, 2, \ldots. \tag{8}$$

□

If the inequality (8) is reversed, the distribution and its survival probabilities are said to be discrete harmonic new worse than used in expectation (discrete HNWUE).

The condition that a distribution is discrete HNBUE is stronger than the condition that the corresponding survival function is HNBUE. In the HNWUE case the situation is just the reverse. The situation for the other aging properties mentioned in the Appendix vary. (Cf. Klefsjö [13].)

In Klefsjö [13] we also prove that the HNBUE (HNWUE) class of life distributions is larger than the NBUE (NWUE) class and that the same is true in the discrete case.

3. RELATIONSHIPS WITH SOME TRANSFORMS

The HNBUE (HNWUE) Property and the Laplace Transform

Let

$$\Phi(s) = \int_0^{\infty} e^{-sx} dF(x) \text{ for } s \geqslant 0 \tag{9}$$

denote the Laplace transform of F. Further let

$$a_k(s) = \frac{(-1)^k}{k!} \cdot \frac{d^k}{ds^k}\left(\frac{1-\Phi(s)}{s}\right) = \int_0^{\infty} \frac{x^k}{k!} e^{-sx} \bar{F}(x)\, dx \text{ for } k = 0, 1, 2, \ldots, \tag{10}$$

and

$$\begin{cases} \alpha_{k+1}(s) = s^{k+1} a_k(s) \text{ for } k = 0, 1, 2, \ldots \\ \alpha_0(s) = 1. \end{cases} \tag{11}$$

Block and Savits [4] proved that a life distribution F is IFR, IFRA, NBU, NBUE, or DMRL (or dual) if and only if $(\alpha_k(s))_{k=0}^{\infty}$ has the corresponding discrete property for every $s > 0$.

The following theorem states that the same is true in the HNBUE and HNWUE cases.

THEOREM 1: A life distribution F is HNBUE (HNWUE) if and only if the sequence $(\alpha_k(s))_{k=0}^{\infty}$ is discrete HNBUE (discrete HNWUE) for every $s > 0$.

PROOF: It is sufficient to prove the theorem in the HNBUE case. The HNWUE case then follows by reversing all inequalities.

The "only if" part in the HNBUE case was proved by Klefsjö [12].

Now, take $t > 0$ and let $s = n/t$. From (10) and (11) it follows that

$$\frac{1}{s}\sum_{k=n+1}^{\infty}\alpha_k(s) = \int_0^{\infty} G_n(x)\ \bar{F}\ (x)dx, \tag{12}$$

where

$$G_n(x) = \sum_{j=n}^{\infty}\frac{(sx)^j}{j!}\ e^{-sx} = \sum_{j=n}^{\infty}\frac{(nx/t)^j}{j!}\ e^{-nx/t}.$$

This means that G_n is a gamma distribution function with the characteristic function

$$h_n(u) = \left(\frac{s}{s - iu}\right)^n = \left(1 - \frac{iut}{n}\right)^{-n}.$$

Since

$$\lim_{n\to\infty} h_n(u) = e^{itu},$$

G_n converges weakly to the one point distribution

$$G(x) = \begin{cases} 0 & x < t \\ 1 & x \geqslant t. \end{cases}$$

Accordingly,

$$\lim_{n\to\infty}\int_0^{\infty} G_n(x)\ \bar{F}(x)dx = \int_0^{\infty} G(x)\ \bar{F}(x)dx = \int_t^{\infty}\bar{F}(x)dx.$$

From this result together with (12) we now obtain that

$$\lim_{n\to\infty}\frac{1}{s}\sum_{k=n+1}^{\infty}\alpha_k(s) = \int_t^{\infty}\bar{F}(x)dx. \tag{13}$$

Since $(\alpha_k(s))_{k=0}^{\infty}$ is discrete HNBUE we get that

$$\sum_{k=n+1}^{\infty}\alpha_k(s) \leqslant (\alpha(s) - 1)\left(1 - \frac{1}{\alpha(s)}\right)^n,$$

where

$$\alpha(s) = \sum_{k=0}^{\infty}\alpha_k(s) = 1 + s\int_0^{\infty}\bar{F}(x)dx = 1 + s\mu.$$

This gives that

$$\frac{1}{s}\sum_{k=n+1}^{\infty}\alpha_k(s) \leqslant \mu\left(1 - \frac{1}{1 + s\mu}\right)^n = \mu\left(1 - \frac{t/\mu}{n + \frac{t}{\mu}}\right)^n. \tag{14}$$

From (13), (14) and the fact that

$$\lim_{n\to\infty}\mu\left(1 - \frac{t/\mu}{n + \frac{t}{\mu}}\right)^n = \mu\ \exp(-t/\mu)$$

we now have

$$\int_t^{\infty}\bar{F}(x)dx \leqslant \mu\ \exp\ (-t/\mu).$$

□

Let ξ_j, $j = 1, 2, \ldots, k$, be independent random variables which are exponentially distributed with $P(\xi_j > x) = \exp(-sx)$, $x \geq 0$. Further, let g_k denote the gamma density function of $\sum_{j=1}^{k} \xi_j$. Then $\alpha_k(s)$ can be written as

$$\alpha_k(s) = \int_0^\infty g_k(x)\, \bar{F}(x)dx \text{ for } k = 1, 2, 3, \ldots .$$

Now suppose that a device is subjected to some kind of shocks and interpret ξ_j as the random damage caused by shock number j. Further, suppose that the damages accumulate additively and that the device fails when the accumulated damage exceeds a random threshold with distribution function F.

Then $\alpha_k(s)$ is the probability that the device will survive the first k shocks. This damage model has been further studied by Esary, Marshall and Proschan [7] and Klefsjö [12], [14].

The Discrete HNBUE (Discrete HNWUE) Property and the Generating Function

Now suppose that $(p_k)_{k=1}^{\infty}$ is a discrete distribution with survival probabilities $\bar{P}_k = \sum_{j=k+1}^{\infty} p_j$, $k = 0, 1, 2, \ldots$, and finite mean $\mu = \sum_{k=0}^{\infty} \bar{P}_k$. A discrete counterpart to the Laplace transform for general survival functions is the generating function

$$k(z) = \sum_{j=1}^{\infty} p_j z^j = 1 - \sum_{j=0}^{\infty} z^j(1 - z)\bar{P}_j.$$

If ξ is a random variable which has a geometric distribution such that

$$P(\xi = j) = z^j(1 - z) \text{ for } j = 0, 1, 2, \ldots \tag{15}$$

we can write $k(z)$ as

$$k(z) = 1 - \sum_{j=0}^{\infty} P(\xi = j)\bar{P}_j.$$

If ξ_ν, $\nu = 1, 2, 3, \ldots, k$, are independent and have the distribution (15) then $\sum_{\nu=1}^{k} \xi_\nu$ has a negative binomial distribution, i.e.,

$$P\left(\sum_{\nu=1}^{k} \xi_\nu = j\right) = \binom{k + j - 1}{j} z^j(1 - z)^k \text{ for } j = 0, 1, 2, \ldots$$

(see Johnson and Kotz [11], p. 124). This means that a discrete counterpart to $(\alpha_k(s))_{k=0}^{\infty}$ in [11] is (with $1 - z$ changed to p and $q = 1 - p$)

$$\begin{cases} \bar{Q}_k(p) = \sum_{j=0}^{\infty} \binom{k + j - 1}{j} p^k q^j \bar{P}_j \text{ for } k = 1, 2, 3, \ldots \\ \bar{Q}_0(p) = 1. \end{cases} \tag{16}$$

Suppose that a device is subjected to two different types of shocks, A and B, say. At every discrete time point a shock of type A occurs with probability p and a shock of type B occurs with probability $q = 1 - p$. If ξ_ν denotes the number of B-shocks between the A-shocks number $\nu - 1$ and ν, $\nu = 1, 2, 3, \ldots$, then ξ_ν has the geometric distribution

$$P(\xi_\nu = j) = pq^j \text{ for } j = 0, 1, 2, \ldots .$$

Further, $\sum_{\nu=1}^{k} \xi_\nu$ represents the number of B-shocks before A-shock number k and

$$P\left(\sum_{\nu=1}^{k} \xi_\nu = j\right) = \binom{k+j-1}{j} p^k q^j \text{ for } j = 0, 1, 2, \ldots .$$

If the device has the probability $\bar{P}_j$ of surviving the first j B-shocks then $\bar{Q}_k(p)$ in (16) denotes the probability that the device will survive until k shocks of type A have occurred.

THEOREM 2: If $(\bar{P}_k)_{k=0}^{\infty}$ is discrete HNBUE (discrete HNWUE) then $(\bar{Q}_k(p))_{k=0}^{\infty}$ defined by (16) is discrete HNBUE (discrete HNWUE) for $0 < p < 1$.

PROOF: Suppose that η is a random variable the distribution of which is a mixture of Poisson distributions, such that the expected values θ of the Poisson distributions vary according to a gamma distribution with probability density function

$$g_p(x) = \left\{\left(\frac{1-p}{p}\right)^k \Gamma(k)\right\}^{-1} x^{k-1} \exp\left(-xp/(1-p)\right) \text{ for } x > 0.$$

Then

$$P(\eta = j) = \int_0^\infty g_p(x)\, e^{-x} \frac{x^j}{j!}\, dx = \binom{k+j-1}{j} p^k q^j$$

(cf. Johnson and Kotz [11], p. 125). This means that, for $k \geqslant 1$, $\bar{Q}_k(p)$ can be written as

$$\bar{Q}_k(p) = \int_0^\infty g_p(x)\, \bar{H}(x)\, dx, \tag{17}$$

where

$$\bar{H}(x) = \sum_{j=0}^{\infty} e^{-x} \frac{x^j}{j!}\, \bar{P}_j. \tag{18}$$

A comparison with (10) and (11) shows that $\bar{Q}_k(p)$ in (17) is of the same form as $\alpha_k(s)$ in (11) but with $s = p/(1-p)$. From Theorem 1 above it follows that $(\bar{Q}_k(p))_{k=0}^{\infty}$ is discrete HNBUE (discrete HNWUE) if and only if $\bar{H}$ in (18) is HNBUE (HNWUE). But Theorem 3.1 in Klefsjö [14] gives that $\bar{H}$ is HNBUE (HNWUE). This proves the theorem. □

REMARK: By using Theorem 1.1 in Block and Savits [4] and Theorems 3.1 and 3.2 in Esary, Marshall and Proschan [7] it follows that Theorem 2 is true with HNBUE (HNWUE) changed to IFR (DFR), IFRA (DFRA), NBU (NWU), NBUE (NWUE) or DMRL (IMRL).

4. PRESERVATION OF THE HNBUE AND HNWUE PROPERTIES UNDER SOME RELIABILITY OPERATIONS

Table 1 shows to what extent the HNBUE and HNWUE classes are closed under some usual reliability operations. For proofs and comments see Klefsjö [13]. Note in particular that

a mixture of HNWUE life distributions is HNWUE. Until recently it was an open question whether the NWUE class had this property. Now Bondesson [5] and Mehrotra [16] independently have proved that the NWUE class is not closed under mixtures. We also mention that a mixture of HNBUE life distributions all of which have the same mean is HNBUE.

TABLE 1 — *A System of Independent HNBUE (HNWUE) Components is HNBUE (HNWUE) According to the Table.*

	Formation of coherent structures	Convolution of Distributions	Mixture of Distributions
HNBUE	Not preserved	Preserved	Not preserved
HNWUE	Not preserved	Not preserved	Preserved

5. BOUNDS ON THE SURVIVAL FUNCTION

In this section we present bounds on the survival function $\bar{F}$ when F is a life distribution which is HNBUE or HNWUE. Such bounds are useful e.g., in reliability since in a typical situation the only fact known a priori may be that the time to failure is HNBUE (HNWUE) with expectation μ.

Bounds of similar character in the IFR (DFR), IFRA (DFRA), NBU (NWU), NBUE (NWUE) and DMRL (IMRL) cases were given by e.g., Barlow and Marshall [1], [2], Marshall and Proschan [15], Haines and Singpurwalla [10] and Barlow and Proschan [3]. Some of these bounds will be presented later on.

Bounds on $\bar{F}$ in the HNBUE Case

Theorems 3 and 4 contain bounds on a HNBUE survival function which are useful when only the mean μ is known.

THEOREM 3: Suppose that F is a life distribution which is HNBUE with mean μ. Then

$$\bar{F}(t) \leqslant \begin{cases} 1 & \text{for } t \leqslant \mu \\ \exp\left(\dfrac{\mu - t}{\mu}\right) & \text{for } t > \mu \end{cases} . \tag{19}$$

PROOF: Let $t > 0$. By using the HNBUE definition (1) we get that

$$\int_s^t \bar{F}(x)\,dx \leqslant \int_s^\infty \bar{F}(x)\,dx \leqslant \mu \exp(-s/\mu) \text{ for every } 0 < s < t.$$

But

$$\int_s^t \bar{F}(x)\,dx \geqslant (t - s)\,\bar{F}(t)$$

since $\bar{F}$ is decreasing. Hence, we obtain that

$$\bar{F}(t) \leqslant \inf_{0<s<t} \frac{\mu \exp(-s/\mu)}{t - s} = \begin{cases} 1 & \text{for } t \leqslant \mu \\ \exp\left(\dfrac{\mu - t}{\mu}\right) & \text{for } t > \mu \end{cases} . \qquad \square$$

We do not know of any analogous upper bounds in the NBUE, NBU and DMRL cases. However, in the IFRA case Barlow and Proschan [3] gave the upper bound

$$\bar{F}(t) \leqslant \begin{cases} 1 & \text{for } t \leqslant \mu \\ \exp(-wt) & \text{for } t > \mu, \end{cases} \tag{20}$$

where $w = w(t) > 0$ is the positive solution to $1 - w\mu = \exp(-wt)$. Calculations show, as is expected, that our HNBUE bound is weaker. The two bounds are illustrated in Figure 1.

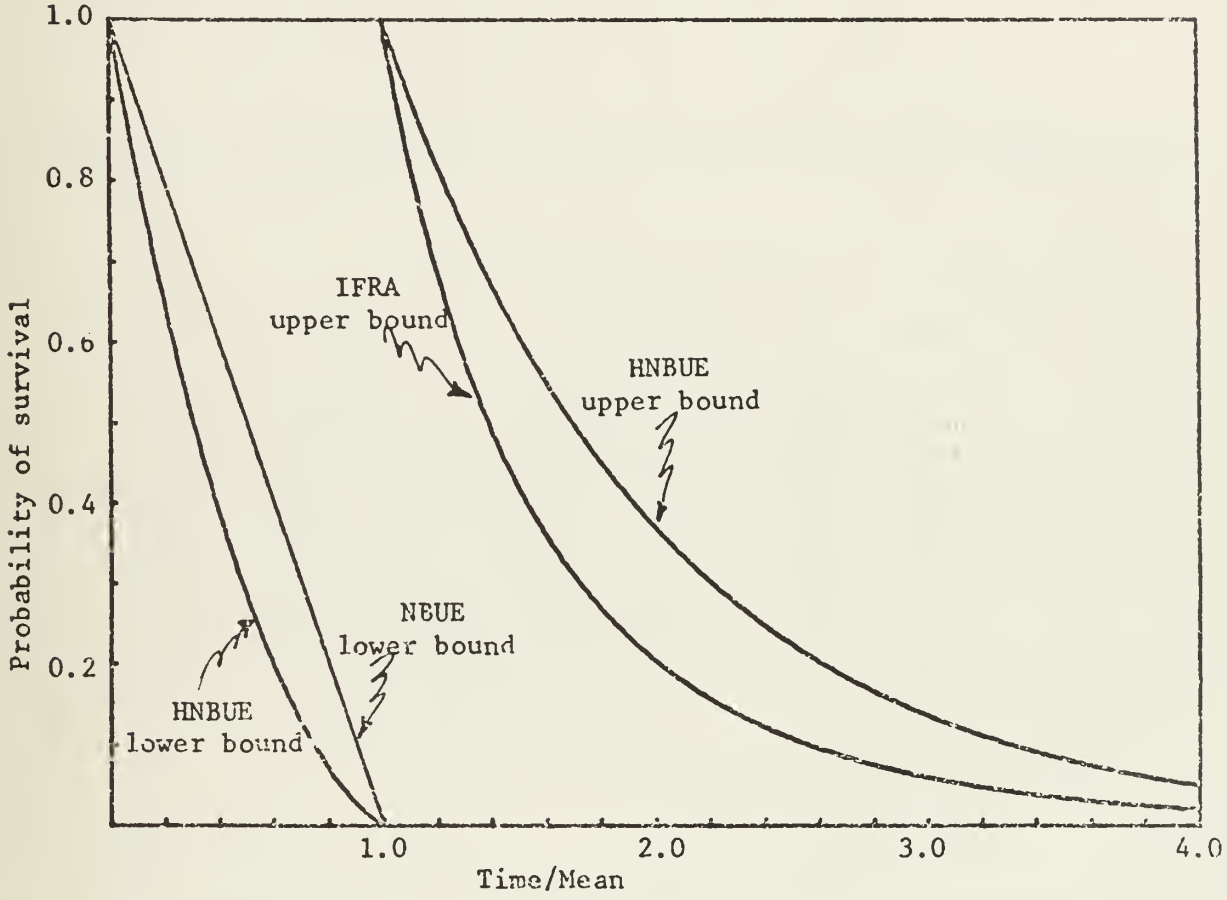

FIGURE 1. The HNBUE upper bound in (19), the IFRA upper bound in (20), the HNBUE lower bound in (21) and the NBUE lower bound in (24).

THEOREM 4: Suppose that F is a life distribution which is HNBUE with mean μ. Then

$$\bar{F}(t) \geqslant \begin{cases} \exp(-\alpha/\mu) & \text{for } 0 \leqslant t < \mu \\ 0 & \text{for } t \geqslant \mu, \end{cases} \tag{21}$$

where $\alpha = \alpha(t)$ is the largest non-negative number for which

$$(\alpha - t + \mu)\exp(-\alpha/\mu) - \mu + t = 0. \tag{22}$$

REMARK: Besides $\alpha = 0$ there is for $0 < t < \mu$ exactly one positive solution to the equation (22).

PROOF: Let $t \geqslant 0$. It follows that

$$\int_0^s \bar{F}(x)dx \leqslant t + \bar{F}(t)\ (s - t) \text{ for every } s > t.$$

Furthermore, the HNBUE definition gives that

$$\int_0^s \bar{F}(x)\ dx \geqslant \mu(1 - \exp(-s/\mu)).$$

Accordingly,

$$\bar{F}(t) \geqslant \{\mu(1 - \exp(-s/\mu)) - t\}/(s - t) \text{ for every } s > t, \tag{23}$$

i.e.
$$\bar{F}(t) \geqslant \sup_{s>t} \{(\mu(1 - \exp(-s/\mu)) - t\}/(s - t).$$

Standard calculus then gives that for $t < \mu$ the supremum is attained for $s = \alpha$ given by (22). Since the right-hand side of (23) is negative for every $s > t$ if $t \geqslant \mu$ the proof is complete. □

Table 2 gives the lower bound in (21) for some values on t/μ.

TABLE 2 — *Some Values of the HNBUE Lower Bound in (21)*

t/μ	lower bound		t/μ	lower bound		t/μ	lower bound
0.00	1.000		0.35	0.509		0.70	0.123
0.05	0.903		0.40	0.446		0.75	0.097
0.10	0.813		0.45	0.334		0.80	0.070
0.15	0.729		0.50	0.285		0.85	0.047
0.20	0.650		0.55	0.239		0.90	0.027
0.25	0.577		0.60	0.198		0.95	0.011
0.30	0.509		0.65	0.161		1.00	0.000

The bound in Theorem 4 can be compared to the result

$$\bar{F}(t) \geqslant \begin{cases} (\mu - t)/\mu & \text{for } t < \mu \\ 0 & \text{for } t \geqslant \mu \end{cases} \tag{24}$$

given by Marshall and Proschan [15] in the NBUE case. Figure 1 illustrates the two bounds.

Bounds on $\bar{F}$ in the HNWUE Case

THEOREM 5: Suppose that F is a life distribution which is HNWUE with mean μ. Then

$$\bar{F}(t) \leqslant \frac{\mu}{t}\ (1 - \exp\ (-t/\mu)) \text{ for } t \geqslant 0. \tag{25}$$

PROOF: The HNWUE definition gives that

$$\int_0^t \bar{F}(x)dx \leqslant \mu\ (1 - \exp(-t/\mu)) \text{ for } t > 0.$$

Since $\bar{F}$ is decreasing

$$\int_0^t \bar{F}(x)\,dx \geqslant t\,\bar{F}(t)$$

and the theorem follows. □

Since the NWUE class of life distributions is the largest of the DFR, DFRA, NWU, NWUE and IMRL classes, all of which are contained in the HNWUE class (cf. Appendix), it may be of interest to compare our HNWUE bound to the NWUE bound

$$\bar{F}(t) \leqslant \frac{\mu}{\mu + t} \text{ for } t \geqslant 0 \tag{26}$$

given by Haines and Singpurwalla [10]. The two bounds are illustrated in Figure 2.

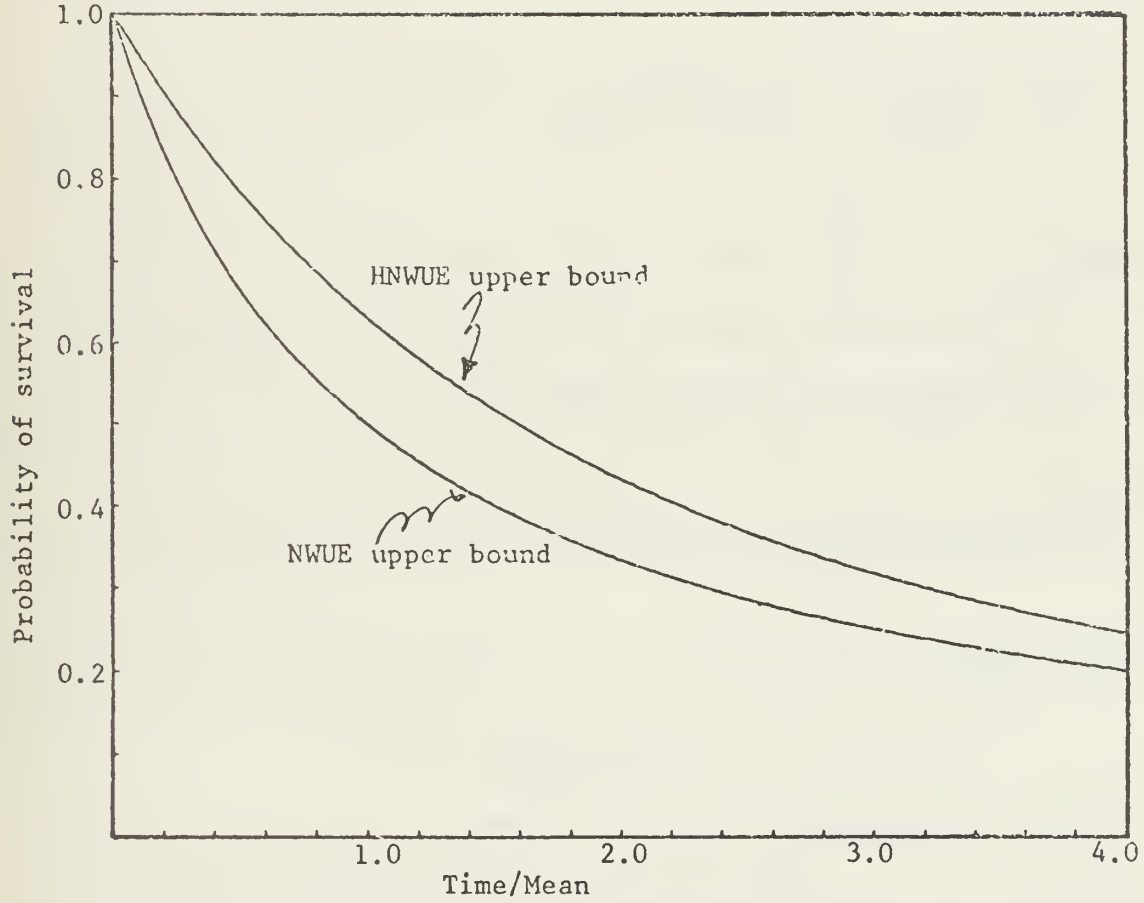

FIGURE 2. The HNWUE upper bound in (25) and the NWUE upper bound in (26)

The only lower bound on $\bar{F}$ we can obtain when $\bar{F}$ is HNWUE is $\bar{F}(t) > 0$. This can be seen by studying $\bar{F}(t) = \epsilon \exp(-\epsilon t/\mu)$, $t > 0$, which is DFR and hence HNWUE.

The bounds presented in Theorems 3, 4, and 5 are all sharp (see Klefsjö [13]).

In Klefsjö [13] bounds are also presented assuming that $\bar{F}(t_*)$ is known for some $t_* > 0$.

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APPENDIX

A life distribution F (i.e. a distribution function with $F(0-) = 0$) and its survival function $\bar{F} = 1 - F$ with support $S = \{t: \bar{F}(t) > 0\}$ and finite mean $\mu = \int_0^\infty \bar{F}(x)\,dx$ are said to be (or to have)

1. increasing failure rate (IFR) if the conditional survival function

$$t \to \frac{\bar{F}(x+t)}{\bar{F}(t)}$$

is a decreasing function of the age t whenever $x > 0$ and $t \in S$;

2. increasing failure rate in average (IFRA) if

$$t \to \frac{-\ln \bar{F}(t)}{t}$$

is increasing on S;

3. new better than used (NBU) if $\bar{F}(x)\bar{F}(y) \geq \bar{F}(x+y)$ for $x \geq 0$ and $y \geq 0$;

4. new better than used in expectation (NBUE) if

$$\bar{F}(x) \int_0^\infty \bar{F}(y)\,dy \geq \int_0^\infty \bar{F}(x+y)\,dy \text{ for } x \geq 0;$$

5. decreasing mean residual life (DMRL) if

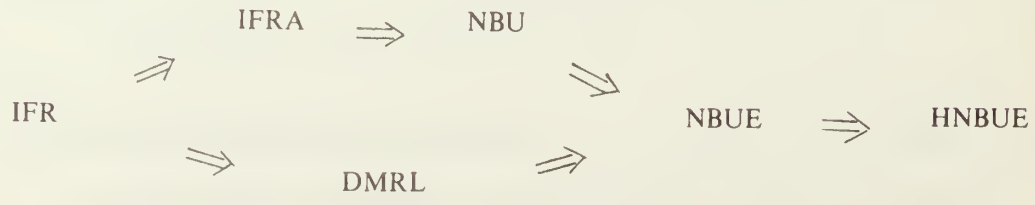
$$t \to \frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(x)\,dx$$

is decreasing on S.

6. harmonic new better than used in expectation (HNBUE) if

$$\int_t^\infty \bar{F}(x)\,dx \leq \mu \exp(-t/\mu) \text{ for every } t \geq 0.$$

The relations between the classes 1.-6. are the following:



By reversing the inequalities and the directions of monotonicity (and studying $t \geqslant 0$) we get the six dual classes DFR, DFRA, NWU, NWUE, IMRL and HNWUE, respectively. Here D = decreasing, I = increasing and W = worse.

A discrete distribution and its survival probabilities $\bar{P}_k = \sum_{j=k+1}^{\infty} p_j$, $k = 0, 1, 2, \ldots$, with $\bar{P}_0 = 1$, finite mean $\mu = \sum_{k=0}^{\infty} \bar{P}_k$ and $S = \{k: \bar{P}_k > 0\}$ are said to be (or to have)

1. discrete increasing failure rate (discrete IFR) if $(\bar{P}_k/\bar{P}_{k-1})_{k=1}^{\infty}$ is decreasing on S;

2. discrete increasing failure rate in average (discrete IFRA) if $(\bar{P}_k^{1/k})_{k=1}^{\infty}$ is decreasing;

3. discrete new better than used (discrete NBU) $\bar{P}_j\bar{P}_k \geqslant \bar{P}_{j+k}$ for $j, k = 0, 1, 2, \ldots$;

4. discrete new better than used in expectation (discrete NBUE) if $\bar{P}_k \sum_{j=0}^{\infty} \bar{P}_j \geqslant \sum_{j=k}^{\infty} \bar{P}_j$ for $k = 0, 1, 2, \ldots$;

5. discrete decreasing mean residual life (discrete DMRL) if $\left(\sum_{j=k}^{\infty} \bar{P}_j\right)/\bar{P}_k$ is decreasing in $k = 0, 1, 2, \ldots$ on S;

6. discrete harmonic new better than used in expectation (discrete HNBUE) if $\sum_{j=k}^{\infty} \bar{P}_j \leqslant \mu(1 - \frac{1}{\mu})^k$ for $k = 0, 1, 2, \ldots$.

The relations between these classes of discrete distributions are the same as in the general case. We get dual classes by reversing the inequalities and directions of monotonicity (and dropping the requirement that $\bar{P}_0 = 1$).

Different properties of the five classes IFR, IFRA, NBU, NBUE and DMRL and their duals are discussed for instance by Bryson and Siddiqui [6], Marshall and Proschan [15], Haines [9] and Barlow and Proschan [3].

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COST ANALYSIS OF PRO RATA AND FREE-REPLACEMENT WARRANTIES

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ABSTRACT

This article examines the short run total costs and long run average costs of products under warranty. Formulae for both consumer cost under warranty and producer profit are derived. The results in the case of the pro rata warranty correct a mistake appearing in Blischke and Scheuer [5]. We also show that expected average cost to both the producer and the consumer of a product under warranty depends on both the mean of the product lifetime distribution and on its failure rate.

1. INTRODUCTION

Manufacturers use warranties to accomplish a variety of goals such as product promotion, quality assurance, and consumer risk reduction. In some cases warranties are required by law; in others cases, though voluntary, warranties are viewed as essential to the producer's competitive position. Consumers, on the other hand, must decide between products with differing warranties or between different warranty options for the same product. The problem of calculating the costs and benefits of warranty policies is an important one.

There are two common types of warranty policies: the pro rata warranty and the free replacement warranty. Both warranties specify a period of time during which the manufacturer will share some responsibility for the costs of repair or replacement of the product. Under a free-replacement policy the producer pays the entire cost of repair or replacement if the product fails before the warranty expires and supplies as many repairs or replacements as needed during the length of the warranty. If the warranty is of a pro rata type, the cost of the replacement item depends on the age of the product at the time of failure and the replacement item is covered by an identical warranty. We assume that after each failure the owner of the product instantaneously purchases an identical replacement. The cost of the replacement depends on the type of warranty and the age of the product at failure. The assumption of instantaneous replacement allows us to describe the sequence of product lifetimes as a renewal process, and to bring to bear on the problem the powerful techniques of renewal theory.

Recently Blischke and Scheuer [5] have approached the problem of calculating the short run costs and benefits to both the buyer and the seller of products covered by warranty policies. Blischke and Scheuer examine both the pro rata and free replacement warranties from the

standpoints of both the buyer and the seller. Previous work by Glickman and Berger [6] examined the free replacement warranty from the standpoint of the seller. Menke [8] also discusses the free replacement type warranty, but assumes that the time between failures of the product is distributed according to an exponential distribution. Bell [3] has examined the long run average profits to the producer of warranty. Beidenweg [4] has also examined the average cost associated with warranties.

In this paper we will formulate a model describing the total cost of ownership of a product under either a pro rata or free replacement type warranty. We will also examine the short run total profit derived from selling a sequence of products covered by either pro rata or free replacement type warranties. The expressions obtained for the cost of ownership and profit under a pro rata warranty correct a mistake appearing in the paper by Blischke and Scheuer [5]. We then derive formulae for the long run average cost of ownership and profit associated with the pro rata and free replacement warranties. These expressions extend partial results obtained by Bell [3]. Finally, we demonstrate that, given two products with equal expected lifetimes, one with a monotone increasing failure rate, and one with a monotone decreasing failure rate, the warranty costs and costs of ownership under a pro rata type warranty will be higher for the increasing failure rate product than for the decreasing failure rate product.

2. SHORT RUN TOTAL COSTS

2.1 Pro Rata Warranty

Suppose that the cost to the seller of replacing the product at the time of failure is constant and equal to q_1. Suppose that the cost to the owner of replacing the product is q_2, where $q_2 > q_1$. The difference between q_2 and q_1 represents the difference between the producer's and consumer's cost of repair or replacement of the product. We assume that the product has only two possible states (working or failed) and that after each failure it is replaced by an identical unit. Thus, there is no distinction between repair and replacement of the product. In what follows we shall speak only of replacement, but the results could apply equally well to the case in which the product is repaired at constant cost after each failure.

Under a pro rata warranty of length W, the owner of the product will pay q_2 if the product fails after time W, and will pay $q_2(X/W)$ if it fails at a time $X \leqslant W$, where X, a random variable, with distribution F, represents the lifetime of the product. We assume that after each failure the customer replaces the product immediately with an identical unit carrying an identical warranty. Under these conditions the sequence of breakdowns is described by a renewal process. We can then write a function $C(X)$, the cost of replacing a unit which survives for a period X before failing:

$$C(X) = \begin{cases} q_2 & \text{if } X > W \\ q_2(X/W) & \text{if } X \leqslant W. \end{cases}$$

Suppose that the owner of the product intends to replace it until time T, after which, should it fail, he will not replace it. T is the "life cycle" or time horizon of the product as defined in Blischke and Scheuer [5]. Let $R(T)$ denote the total replacement costs incurred from owning the product up to time T, ignoring the initial purchase cost. We wish to compute the total expected replacement costs up to time T, namely $E[R(T)]$. Conditioning on the time

of the first breakdown, X_1, and using the fact that under a pro rata warranty the warranty is renewed after each failure, we may write:

$$E[R(T)|X_1 = x] = \begin{cases} 0 & \text{if } x > T \\ q_2 + E[R(T-x)] & \text{if } W < x \leqslant T \\ q_2(x/W) + E[R(T-x)] & \text{if } 0 < x \leqslant W \end{cases}$$

By the law of total probability:

$$E[R(T)] = \int_W^T [q_2 + E[R(T-x))]\, dF(x) + \int_0^W [q_2(x/W) + E[R(T-x))]\, dF(x).$$

Let $\mu_w = \int_0^W x dF(x)$. Define:

$$h(t) = \int_0^t (q_2 \wedge (q_2 x/W))\, dF(x).$$

We can rewrite the above equation as:

$$E[R(T)] = h(T) + \int_0^T E[R(T-x)]\, dF(x).$$

This is a standard renewal equation and has solution:

$$E[R(T)] = h(T) + \int_0^T h(T-u)\, dm(u).$$

Where in the above expression $m(x)$ is the renewal function associated with the distribution F. The renewal function is defined by the equation:

$$m(x) = \sum_{n=1}^{\infty} F^{(n)}(x)$$

where $F^{(n)}(x)$ represents the n-fold convolution of F evaluated at the point x. We can rewrite the expression for $E[R(T)]$ as:

$$(1) \qquad E[R(T)] = q_2[F(T) - F(W) + \mu_w/W] + q_2 \int_0^{T-W} F(T-x)\, dm(x) + \int_{T-W}^T \int_0^{T-x} (q_2 z/W)\, dF(z)\, dm(x).$$

Denote by $E[TC_{PR}(T)]$ the total costs of replacement during the period $[0,T]$, then:

$$(2) \qquad E[TC_{PR}(T)] = q_2 + E[R(T)] = q_2 + q_2[F(T) - F(W) + \mu_w/W] + q_2 \int_0^{T-W} F(T-x)\, dm(x) + \int_{T-W}^T \int_0^{T-x} (q_2 z/W)\, dF(z) dm(x).$$

We can derive the seller's profit during the product life cycle by noting that the consumer's cost is equal to the producer's revenue. The producer's profit is then his revenue less the costs of the replacements made in the interval $(0,T]$. The cost of the replacements is $q_1 m(T)$. Thus, the seller's profit is:

$$(3) \qquad E[\pi_{PR}(T)] = (q_2 - q_1) + E[R(T)] - q_1 m(T).$$

Equation (2) disagrees with the result presented in the paper by Blischke and Scheuer [5]. Using the notation of this paper we can write the expression derived in [5] for the cost of repair of a single failed item:

$$(4) \qquad (q_2/W)\, [\mu_w + W(1 - F(W))].$$

Blischke and Scheuer condition on the number of failures during the period $[0,T]$. They then take expectations and obtain:

$$q_2 + m(T)\ (q_2/W)\ [\mu_w + W(1 - F(W))]. \tag{5}$$

Conditioning on the number of failures in an interval changes the distribution of time between failures in that interval, invalidating equation (4). Thus the result of Blischke and Scheuer for the pro rata warranty is in error. An expression similar to (1) has also been obtained by Biedenweg [4] but he does not mention the flaw in the argument of Blischke and Scheuer.

2.2 The Free-Replacement Warranty

The analysis of the free-replacement warranty is more complex than that of the pro rata warranty. The cost to the consumer of replacement is zero if the product fails before the end of the warranty period, W. If the product fails after time W then the consumer pays the full price for the replacement. Any replacement made during the warranty period is warrantied only until the end of the period, thus the warranty is not renewed after each replacement. When the product fails outside of the warranty period and the consumer replaces it at full cost, then the replacement item is covered by a free replacement warranty identical to the warranty on the original item purchased. This guarantees that the consumer will receive as many free replacements as needed during the warranty period, but must buy the replacements at full cost after the warranty period. Thus, the consumer pays every $W + e(W)$ units of time, where $e(W)$ is the excess life of the process $X_1, X_2, \ldots$ at time W. If $N(W)$ represents the number of failures by time W, and S_n represents the sum of the first n lifetimes, then the random variable $e(W)$ is given by:

$$e(W) = S_{N(W)+1} - W.$$

Let $C_{FR}(T,W)$ represent the expected cost of owning a product with a warranty of length W for a length of time T exclusive of initial purchase cost. If, as before, we condition on the time until the first failure, X_1, we get the following equation for $E[C_{FR}(T,W)|X_1 = x]$:

$$E[C_{FR}(T,W)|X_1 = x] = \begin{cases} 0 & \text{if } x > T \\ q_2 + E[C_{FR}(T - x,\ W)] & \text{if } W < x \leqslant T \\ E[C_{FR}(T - x,\ W - x)] & \text{if } 0 < x \leqslant W \end{cases}$$

This yields the following functional equation for $E[C_{FR}(T,\ W)]$:

$$E[C_{FR}(T,W)] = q_2(F(T) - F(W)) + \int_W^T E[C_{FR}(T - x,\ W)]\ dF(x) + \int_0^W E[C_{FR}(T - x,\ W - x)]\ dF(x). \tag{6}$$

The total expected cost of ownership during the period $[0,T]$, of a product under a free-replacement warranty, $TC_{FR}(T,W)$ is given by:

$$TC_{FR}(T,W) = E[C_{FR}(T,W)] + q_2.$$

We can determine a priori that $TC_{FR}(x,y) = q_2$ if $y < x$.

We can derive an expression for the expected total profit earned during the period $[0,T]$. Let $P_{FR}(T,W)$ represent the profit, exclusive of the profit on the sale at time 0, earned during $(0,T]$. As, before, by conditioning on the length of the lifetime of the first product we get an expression for $E[P_{FR}(T,W)|X_1 = x]$:

$$E[P_{FR}(T,W)|X_1 = x] = \begin{cases} 0 & \text{if } x > T \\ q_2 - q_1 + E[P_{FR}(T-x,\ W)] & \text{if } W < x \leqslant T. \\ E[P_{FR}(T-x,\ W-x)] - q_1 & \text{if } 0 < x \leqslant W \end{cases}$$

The expected profit in $(0,T]$ is given by:

$$(7) \qquad E[P_{FR}(T,W)] = q_2[F(T) - F(W)] - q_1[F(T)] + \int_W^T E[P_{FR}(T-x,\ W)]dF(x) + \int_0^W E[P_{FR}(T-x,\ W-x)]\,dF(x).$$

Expected total profit in the period $[0,T]$ is given by:

$$E[TP_{FR}(T,W)] = E[P_{FR}(T,W)] + q_2 - q_1.$$

Equations (6) and (7) are difficult to solve in general. Blischke and Scheuer compute the expected profit only for the case in which $T = W$, and approximate the solution to (6) by:

$$TC_{FR}(T,W) \approx q_2(1 + m_Y(T)) \approx q_2(1 + T/(E(X)[1 + m(W)])$$

where $m_y(T)$ is defined to be the renewal function of the random variable $Y = W + e(W)$. They justify this approximation by observing that as $T \to \infty$, $m_Y(T)/T \to 1/E(X)[1 + m(W)]$.

3. LONG RUN AVERAGE COST

In many situations the long run averages of cost and profit are better criteria for comparing alternative policies. This is particularly true in situations in which the product life-cycle is expected to be quite long. In this section we will derive exact formulas for the long run average cost and profit form a product under three possible conditions, a no warranty, a pro rata warranty, and a free replacement warranty. We will make use of the following result which is proved in Johns and Miller [7] and Ross [9].

Suppose that the sequence of pairs of random variables (Y_i, X_i), $i = 1, \ldots$ form an independent and identically distributed sequence, then:

$$1)\ \lim_{T\to\infty} 1/T \sum_{i=1}^{N(T)} Y_i = E(Y_i)/E(X_i) \text{ with probability 1}$$

and:

$$2)\ \lim_{T\to\infty} (1/T)E\left[\sum_{i=1}^{N(T)} Y_i\right] = E(Y_i)/E(X_i).$$

In the above equations $N(t)$ is the random variable representing the number of renewals of the process (X_i), $i = 1, \ldots$ in the interval $(0,T]$. The above equations imply that the almost sure limit and the limit of the expected value of $(1/T)\sum_{i=1}^{N(T)} Y_i$ are the same. In what follows we shall use "average" to mean either of these quantities; since they are the same no ambiguity will arise.

3.1 The No Warranty Case

In the case in which the sequence of products purchased by the consumer is not covered by a warranty the analysis of the long run average cost is quite simple. The consumer pays an

amount q_2 after every failure. Since the sequence of product lifetimes, $X_1, X_2, \ldots$ was assumed to be a sequence of independent and identically distributed random variables; (q_2, X_i) comprise a renewal reward process, and the average cost of ownership, AC_{NW}, is given by:

$$AC_{NW} = q_2/E(X).$$

Similarly the seller receives a profit of $q_2 \cdot q_1$ every X_i units of time yielding and average profit, AP_{NW} of:

$$AP_{NW} = (q_2 - q_1)/E(X).$$

Since the cost, q_2, and profit, $q_2 - q_1$, of the initial purchase are constant, and finite, they do not change the long run average cost and profit. Hence, we may ignore them.

3.2 The Pro Rata Warranty

The long run average cost of ownership of a product under a pro rata warranty may be computed from equation (3). An easier approach is to note that the cost of replacement of a product under a pro rata warranty is independent of prior and future replacement costs. As in Section 2.1 denote the cost of the i^{th} product purchased by $C(X_i)$. $(C(X_i), X_i)$ comprise a renewal reward process. The long run average cost of replacement is given by:

$$\lim_{T\to\infty} 1/T \sum_{i=1}^{N(T)} C(X_i) = E[C(X_i)]/E(X_i) = (q_2)\,[\mu_W + W(1 - F(W))]/WE,\,(X). \tag{8}$$

We can develop a more convenient expression for the average cost of ownership under a pro rata warranty in terms of the cumulative hazard function of the distribution of the lifetime of the product. Observe that since $F(0) = 0$:

$$\mu_W = \int_0^W x dF(x) = \int_0^W (1 - F(x))\,dx - (1 - F(W))\,W.$$

If we substitute this formula into equation (8) we get, after simplifying:

$$AC_{PR}(W) = (q_2/W) \int_0^W (1 - F(x))\ dx/E(X).$$

Assume that F has a density and denote the failure rate of F by $h(x)$. Let $H(x)$ represent the cumulative failure rate up to time x. The failure rate, $h(x)$ is given by:

$$h(x) = f(x)/[1 - F(x)].$$

F is said to have the increasing failure rate property (F is IFR) if $h(x)$ is increasing and the decreasing failure rate property if $h(x)$ is decreasing (F is DFR). $H(x)$ is given by:

$$H(x) = \int_0^x h(u)\,du.$$

We can now make use of the following formula for $1 - F(x)$ (see Barlow and Proschan [1]):

$$1 - F(x) = e^{-H(x)}dx.$$

We can write the average cost of ownership of a product under a pro rata warranty as:

$$AC_{PR}(W) = (q_2/W)\int_0^W e^{-H(x)}\,dx/E(X). \tag{9}$$

A similar calculation yields an expression for the average profit to the seller of a product under a pro rata warranty of length W. Denote the average profit by $AP_{PR}(W)$:

$$AP_{PR}(W) = [(q_2/W)\int_0^W e^{-H(x)}\,dx - q_1]/E(X). \tag{10}$$

Note that the average cost and profit depend on both the cumulative hazard function and on the mean of the distribution of product lifetime. We can illustrate this dependence in the following:

PROPOSITION: Consider two different products with lifetimes described by the random variables X_1 and X_2 with common expectation μ. Let F_1 be the distribution of X_1 and F_2 the distribution of X_2. Suppose that the replacement cost to the consumer is identical for each product. If F_1 has a continuous increasing failure rate and F_2 has a continuous decreasing failure rate then:

$$AC^1_{PR}(W) \geqslant AC^2_{PR}(W).$$

Where $AC^1_{PR}(W)$ is the long run average cost of ownership of product 1 and $AC^2_{PR}(W)$ is the longrun cost of ownership of product 2.

PROOF: Since $AC^1_{PR}(W) - AC^2_{PR}(W)$ is given by:

$$(q_2/\mu W)\left[\int_0^W e^{-H_1(x)}dx - \int_0^W e^{-H_2(x)}dx\right].$$

It is sufficient to show that, for all w:

$$\int_0^w e^{-H_1(x)}\,dx \geqslant \int_0^w e^{-H_2(x)}\,dx.$$

From Barlow and Proschan [1] p. 113, we know that, if F_1 is *IFR* and F_2 is *DFR*, then for any $t < \mu$:

$$e^{-H_1(t)} \geqslant e^{-H_2(t)}$$

Hence for any $t < \mu$:

$$\left[\int_0^t e^{-H_1(x)} - e^{-H_2(x)}\,dx\right] \geqslant 0$$

Now suppose that at some point $z > \mu$,

$$\left[\int_0^z e^{-H_1(x)} - e^{-H_2(x)}\,dx\right] < 0.$$

Since $e^{-H_1(x)} - e^{-H_2(x)}$ is continuous there must be some interval beyond μ on which it is negative.

But by assumption:

$$\int_0^\infty e^{-H_1(x)}\,dx = \int_0^\infty e^{-H_2(x)}\,dx.$$

So there must exist and interval beyond z on which $e^{-H_1(x)} - e^{-H_2(x)}$ is positive. These two observations, combined with the assumption of the continuity of $e^{-H_1(x)} - e^{-H_2(x)}$ establish that there must exist x_1 and x_2, $\mu < x_1 < x_2 < \infty$ such that:

$$e^{-H_1(x_1)} - e^{-H_2(x_1)} = 0$$
$$e^{-H_1(x_2)} - e^{-H_2(x_2)} = 0.$$

Thus:

$$H_1(x_1) - H_2(x_1) = 0$$
$$H_1(x_2) - H_2(x_2) = 0.$$

Since it is also true that $H_1(0) - H_2(0) = 0$, we can apply the mean value theorem to $H_1(x) - H_2(x)$ between 0 and x_1 and between x_1 and x_2 to find two points r_1 and r_2, $r_1 < x_1 < r_2 < \infty$ such that:

$$H_1'(r_1) - H_2'(r_1) = h_1(r_1) - h_2(r_1) = 0$$
$$H_1'(r_2) - H_2'(r_2) = h_1(r_2) - h_2(r_2) = 0.$$

But the last two equalities indicate that h_1 and h_2 cross twice, contradicting the assumption of monotone failure rates. This establishes the proposition. If we assume that the failure rates are only monotone, rather than monotone and continuous, $h_i(x)$ will be continuous except at a set of points of Lesbegue measure 0. Hence, $H_i(x)$ will be differentiable everywhere except possibly at a set of points of measure 0. The proof can be generalized to include this case. A similar proposition may be established for the average profit.

3.3 The Free-Replacement Warranty

As before, let $e(W)$ denote the excess life of the renewal process $X_1, X_2, \ldots$ at time W. Under a free replacement warranty of length W the buyer need only pay for a replacement after every $w + e(W)$ units of time. At these points in time the sequence of payments regenerates. Define Y_1, Y_2, to be a sequence of random variables which are independent with distributions identical to the distribution of $W + e(W)$. Y_i represents the time between the i^{th} and $i + 1^{\text{st}}$ time that the customer must pay for a replacement. We shall call Y_i the i^{th} "warranty cycle." The sequence (Y_i, q_2) forms a renewal reward process. Hence, the average cost of ownership of the product under a free replacement warranty is:

$$AC_{FR}(W) = q_2/E(Y) = q_2/(m(W) + 1)\,E(X). \tag{11}$$

Again, $m(W)$ is the renewal function of the common distribution, F of the sequence X_i. As in the case in which there is no warranty, the initial cost has no effect on the long-run average cost.

The seller of the product replaces the product for free whenever it fails during the warranty period, incurring a cost q_1. At the first time it fails outside of the warranty period the seller nets $q_2 - q_1$. Let $Q(Y)$ represent the net revenue received from the sale of replacements during a single warranty cycle. The sequence $(Y_i, Q(Y_i))$ forms a renewal reward process. We can find $E[Q(Y_i)]$ by solving equation (7) for the case in which $T = W$. It is simpler, however, to note that the expected profit in any warranty period is equal to the difference between the expected cost of replacements during the warranty period, and the profit from the first replacement after the warranty period is given by:

$$E[Q(Y)] = (q_2 - q_1) - q_1 m(W).$$

The long run average profit is given by:

$$AP_{FR}(W) = E[Q(Y)]/E(Y) = [(q_2 - q_1) - q_1\, m(W)]/(m(W) + 1)\, E(X), \tag{12}$$

If we now multiply the long run average cost of ownership under warranty by the length of the life cycle T, we will have an estimate of the short run total cost up to time T:

$$TC_{FR}(T, W) \approx T[q_2/(m(W) + 1)\, E(X)].$$

This formula is similar to the estimate used by Blischke and Scheuer. It is clear from our discussion and from the theory of renewal reward processes that (see Johns and Miller [7]):

$$\{TC_{FR}(T, W) - T[q_2/(m(W) + 1)\, E(X)]\}\,/\,T \to 0 \text{ as } T \to \infty.$$

4. A NUMERICAL EXAMPLE

In this section we will illustrate the results of the previous section for three different product lifetime distributions. We will consider the gamma, exponential, and hyperexponential distributions. The gamma density is given by:

$$f_{\alpha,\lambda}(x) = \lambda^{\alpha}/[\Gamma(\alpha)]x^{\alpha-1}\, e^{-\lambda x}\; x \geqslant 0,\; \alpha > 0,\; \lambda > 0.$$

In the above equation $\Gamma(\alpha)$ is the gamma function. The gamma distribution has mean α/λ. For $\alpha > 1$ it has a monotone increasing failure rate and for $\alpha < 1$ it has a montone decreasing failure rate (see Barlow and Proschan [1] p. 75). The exponential density is given by:

$$g_{\delta}(x) = \delta e^{-\delta x}\; x \geqslant 0,\; \delta > 0.$$

The mean of the exponential distribution is $1/\delta$. The exponential distribution has a constant failure rate. The hyperexponential density is given by:

$$h_{\gamma,\beta_1,\beta_2}(x) = \gamma\beta_1 e^{-\beta_1 x} + (1 - \gamma)\beta_2 e^{-\beta_2 x}.$$

The hyperexponential distribution has mean $\gamma(1/\beta_1) + (1 - \gamma)\ (1/\beta_2)$ and a decreasing failure rate. Tables 1 and 2 display the long run average cost to the consumer and profit to the producer for pro rata and free replacement warranties of various lengths. For the gamma distribution the parameters were taken to be $\alpha = 1.5$, $\lambda = 2$, for the exponential, $\delta = 1.333$ and for the hyperexponential, $\gamma = .5$, $\beta_1 = 8$ and $\beta_2 = .7273$. In each case this yields a mean life of .75 (we take the time unit to be years). Thus, we have three different product lifetime distributions, with the same mean, with increasing (gamma), constant (exponential), and decreasing (hyperexponential) failure rates. For the cases described in Table 1 and 2 the producer's and consumer's costs of repair were set equal to \$1. Thus, the entries for the producer's profit in the tables are negative.

The results of Table 1 are consistent with the behavior predicted by the proposition of Section 3. The entries in the table were computed using formulae (9) and (10). For each warranty length the cost to the consumer is higher for the gamma distribution than for either the exponential or hyperexponential distributions. The difference in cost to the consumer rises quite sharply for relatively short warranty lengths. For a warranty length of .6 years the long run average cost to the consumer for a product with the gamma distributed lifetime is 48% higher than for a product with the hyperexponentially distributed lifetime. The difference decreases as the length of the warranty increases until, for a warranty length of four years, the average cost to the consumer is only 5% higher for the gamma distributed lifetime. This behavior is intuitive as the average cost to the consumer from a pro rata warranty approaches 0 as the warranty length increases, and $1/E(X)$ as the warranty length approaches 0, regardless of the product lifetime distribution.

From the proposition of Section 3 and Table 1 we can see that the common practice of estimating a product lifetime distribution with known mean by an exponential distribution with the same mean may lead to serious under or over estimates of average cost and profit under warranty. This result is particularly important in the case in which the product lifetime distribution has a decreasing failure rate. Decreasing failure rate distributions are often associated with products which suffer from "infant mortality," i.e., a small proportion of products fail early. This is precisely the situation in which a warranty policy may be useful. However, the use of an exponential distribution to approximate the product lifetime distribution will yield an over estimate of long run average costs of ownership under warranty. For the case illustrated in Table 1, at a warranty length of .6 years the exponential distribution yields a long run average cost of ownership which is 35% higher than for the hyperexponential distribution. For a length of 4 years the difference is 5%.

The calculation of the costs and benefits of the free replacement warranty is somewhat more difficult, as the formulae for $AC_{FR}(W)$ and $AP_{FR}(W)$ (equations (11) and (12)) involve the renewal function explicitly. Fortunately, it may be calculated easily for the exponential and hyperexponential distributions. For the gamma distribution the renewal function has been tabulated. The tables of Baxter, Scheuer, Blischke and McConalogue [2] were used in the preparation of Table 2.

Table 2 shows much the same pattern as Table 1. The difference is long run average cost between gamma and hyperexponential distributions does not fall as quickly as it does for the pro rata warranty. From the results of Tables 1 and 2 we can find the warranty lengths for the pro rata and free-replacement warranties which yield equal long run average costs of ownership. We can see that the pro rata warranty required to reduce costs to a given level is longer than the free replacement warranty needed to reduce average costs to the same level.

5. CONCLUSION

In this paper we have derived expressions for the short run total cost of ownership and short run total profit from the sale of a sequence of products under a pro rata warranty. Equation (2) corrects a mistake in the literature. We develop an integral equation for the short run total cost and profit under a free replacement warranty. Throughout our derivation we have taken the approach of computing the total cost of ownership exclusive of initial purchase cost, and then adding in the initial purchase cost. This approach has the advantage of allowing us to treat, with minor changes in notation, the situation in which the cost of the first item sold is different from the cost of subsequent items.

TABLE 1 — *Long Run Average Cost-Pro Rata Warranty*

Warranty Length (year)	Gamma		Exponential		Hyperexponential	
	Consumer Cost ($)	Producer Profit ($)	Consumer Cost ($)	Producer Profit ($)	Consumer Cost ($)	Producer Profit ($)
.2	1.248	−.086	1.170	−.163	.953	−.380
.4	1.126	−.207	1.033	−.300	.778	−.555
.6	1.005	−.328	.918	−.416	.678	−.655
.8	.895	−.437	.820	−.514	.609	−.724
1.0	.799	−.534	.736	−.597	.557	−.776
1.2	.715	−.618	.665	−.668	.514	−.819
1.4	.643	−.690	.604	−.729	.478	−.856
1.6	.582	−.752	.551	−.782	.446	−.887
1.8	.529	−.805	.505	−.828	.418	−.915
2.0	.483	−.850	.465	−.868	.393	−.940
2.2	.444	−.889	.430	−.903	.370	−.963
2.4	.410	−.923	.400	−.934	.350	−.983
2.6	.380	−.953	.373	−.961	.331	−1.002
2.8	.354	−.979	.349	−.985	.314	−1.019
3.0	.331	−1.002	.327	−1.006	.299	−1.034
3.2	.311	−1.022	.308	−1.025	.285	−1.049
3.4	.293	−1.050	.291	−1.042	.271	−1.062
3.6	.277	−1.056	.275	−1.058	.259	−1.074
3.8	.263	−1.071	.261	−1.085	.248	−1.085
4.0	.250	−1.084	.249	−1.072	.238	−1.096

TABLE 2 — *Long Run Average Cost-Free Replacement Warranty*

Warranty Length (year)	Gamma		Exponential		Hyperexponential	
	Consumer Cost ($)	Producer Profit ($)	Consumer Cost ($)	Producer Profit ($)	Consumer Cost ($)	Producer Profit ($)
.2	1.151	−.183	1.053	−.281	.798	−.535
.4	.958	−.375	.870	−.464	.633	−.700
.6	.811	−.523	.741	−.593	.546	−.788
.8	.684	−.650	.645	−.688	.487	−.847
1.0	.615	−.719	.571	−.762	.442	−.892
1.2	.548	−.786	.513	−.820	.405	−.928
1.4	.494	−.840	.465	−.868	.375	−.959
1.6	.450	−.884	.426	−.908	.348	−.985
1.8	.412	−.921	.392	−.941	.326	−1.008
2.0	.381	−.952	.364	−.970	.306	−1.028
2.2	.354	−.979	.339	−.994	.288	−1.045
2.4	.331	−1.003	.317	−1.016	.272	−1.061
2.6	.310	−1.023	.299	−1.035	.258	−1.075
2.8	.292	−1.041	.282	−1.052	.246	−1.088
3.0	.276	−1.057	.267	−1.067	.234	−1.099
3.2	.261	−1.072	.253	−1.080	.224	−1.110
3.4	.248	−1.085	.241	−1.092	.214	−1.119
3.6	.237	−1.097	.230	−1.103	.205	−1.128
3.8	.225	−1.108	.220	−1.114	.197	−1.136
4.0	.216	−1.117	.211	−1.123	.190	−1.144

We also derive formulae for the long run average cost and profit in the case of the pro rata and free replacement warranties. It is possible to get exact, closed form expressions for average cost and profit for both the pro rata and free replacement warranties, a definite improvement over the short run case. The formulae for AC_{NW}, AC_{PR}, and AC_{FR} can be used to compare the costs to the buyer of no warranty, a pro rata warranty and a free replacement warranty. Similarly AP_{NW}, AP_{PR}, and AP_{FR} can be used to compare the profitability of the various types of warranty policies. Our approach to average profits is similar to that of Bell [3]. The expressions also allowed us to see explicitly in the case of the pro rata warranty the relationship between hazard rate and the cost of ownership under warranty.

ACKNOWLEDGMENTS

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THE OUTFIT PLANNING PROBLEM: PRODUCTION PLANNING IN SHIPBUILDING*

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ABSTRACT

Shipbuilding as currently practiced in U.S. commercial shipyards employs little quantitative modeling or analysis in production planning. This paper presents a brief discussion of the shipbuilding process and focuses on one major component which is referred to as *outfitting*. The outfit planning problem is described in detail and then formally modeled as a generalization of the resource constrained project scheduling problem. The value of the approach as well as barriers to its adoption are also discussed.

1. INTRODUCTION

For a number of years, the U.S. shipbuilding industry as a whole has been recognized as not competitive with the best foreign shipbuilders. Because shipbuilding is considered a vital industry, commercial shipyards at one time received a substantial subsidy, which was administered by the Maritime Administration (MarAd) of the Department of Commerce. In addition, MarAd supports various research projects through the Shipbuilding Research Program, which are aimed at improving shipbuilding in U.S. shipyards.

In the past, MarAd's research emphasis has been technologically or design oriented, e.g., welding technology, cutting technology, propeller design, hull form design and so forth. More recently, there has been a growing realization that important problems also exist in production methodology, including work methods, production standards, production planning and production control. This paper addresses a particular problem in production planning and control in shipyards, and presents some of the findings of a MarAd sponsored research project.

The problem considered in this paper is called the *outfit planning problem.* Outfitting refers to the fabrication and installation in a ship of everything that is not considered hull steel, i.e.,

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everything except the hull itself, the decks, and the bulkheads. In many instances, outfitting represents as much as 50% of the cost of the ship and also as much as 50% of the elapsed time for production. Clearly, any improvement in the outfitting process would have a significant impact on the shipbuilding industry.

The primary contributions of this paper are:

a. To describe this important class of production planning problems;

b. To present a formal model of the problem;

c. To examine the impact of shipyard's adopting the solution approach.

1.1 The Shipbuilding Process*

Shipyard production activities can be broken into two distinct groups: the *steel phase activities*, which encompass all the activities associated with fabricating and assembling the hull, up to and including complete ship erection; and the *outfit phase activities,* which are associated with the acquisition, fabrication and installation in the ship of everything else.

Conceptually at least, the outfit phase activities could all come after the steel phase activities were completed. As a practical matter, of course, this would not be a feasible production method because of the expense of opening up closed compartments to land equipment or to install outfit material such as piping. This extreme oversimplification does, however, capture much of the traditional concept of outfitting as a "successor function" [6]. That is, production often has been treated as two distinct phases with very little interfacing of the steel and outfit activities.

Although various production methods are practiced in U.S. shipyards, the most common method is *hull block construction.* The ship is divided geographically into components, or blocks. A typical ship might consist of 100 blocks. The steel components of the block are assembled in a block assembly area, and the block, perhaps weighing as much as several hundred tons, is then lifted onto the ways for final erection.

Shipbuilding can be viewed in terms of the material flows and primary production facilities. As shown in Figure 1, ship production occurs in three primary facilities with two major categories of supporting facilities plus outside vendors. The steel shops represent facilities where the steel forming activities take place. This includes welding of stiffeners and bracing to large steel plates. Similarly "other shops" include all the facilities associated with fabricating sheet metal, ducting, wire, piping, equipment, etc.

In this model, the "assembly area" represents any configuration of facilities where steel and/or other materials are brought together and processed prior to actual ship erection, i.e., prior to going on the ways. The ways area is the facility where ship erection, i.e., hull assembly, takes place. After hull assembly, the ship is completed at least to the point of being able to float. The "outfit pier" represents the stage of ship production which follows float-off.

*Reference [3] provides a good summary of current practices in U.S. shipyards.

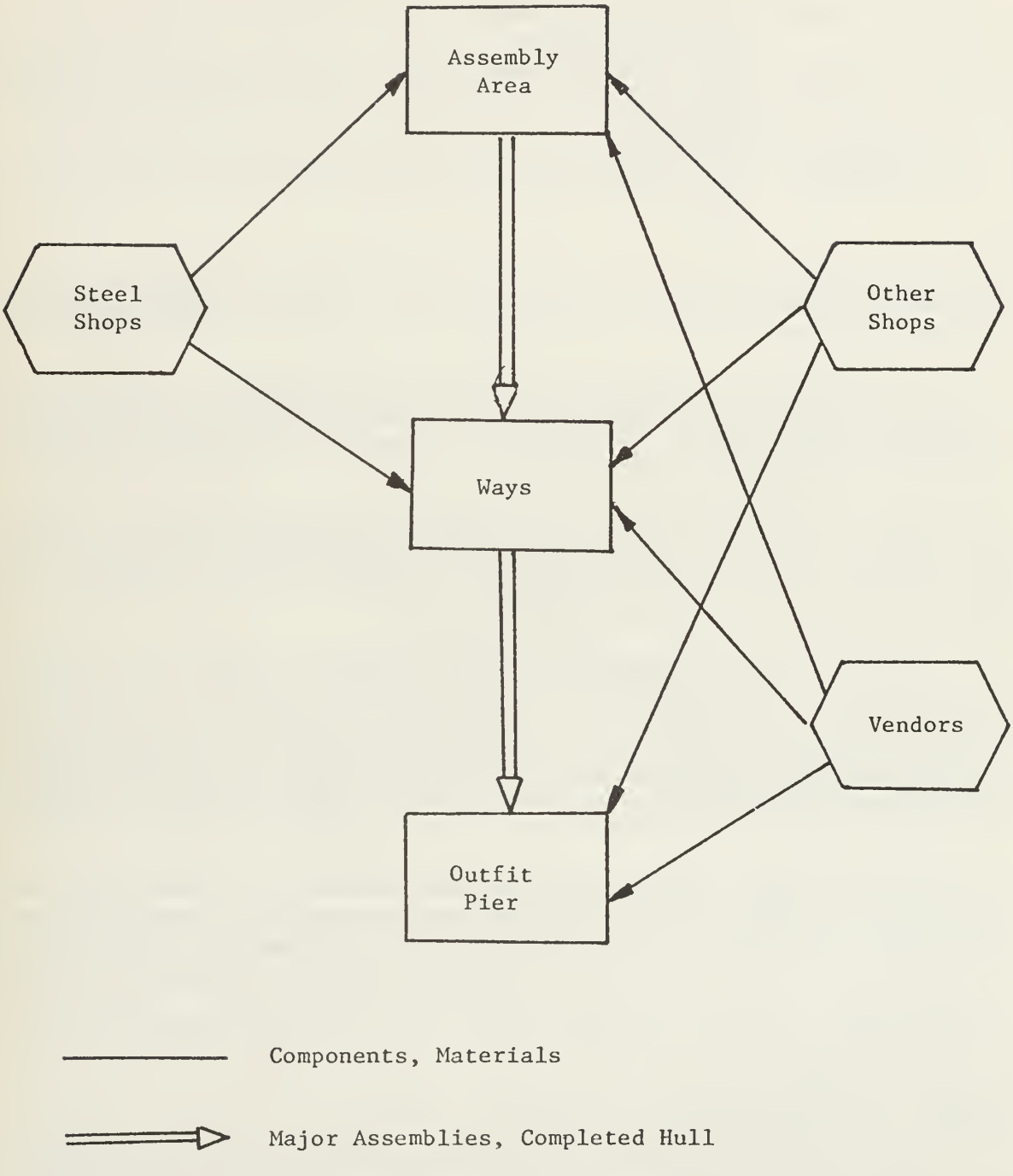

FIGURE 1. General production flow model

The consideration of facilities and material flows leads naturally to the idea of different production *modes*:

(1) fabrication: the production of individual pieces of steel, sheet metal, ducting, piping, electrical cable, etc.
(2) assembly: fabricated plates are assembly to form *blocks;* also individual components may be assembly to form *units*, e.g., equipment with foundations, valve with piping, etc.
(3) erection: the activity on the ways that results in the completed hull.
(4) outfitting: the remaining production activities that take place once erection is completed.

Steel phase activities cannot be performed after erection, by definition. Outfitting phase activities, however, can be performed in any of the four modes. If they occur before erection, they are *preoutfit* activities.

Note that the terms "steel phase" and "outfitting phase" have been used to delineate activities by *type*. The four modes defined above, however, delineate activities by the timing of performance and facilities required. The distinction is an important one since there are options for many activities with regard to production mode.

There are two important aspects to production planning as generally practiced in hull block construction. The first is the *work breakdown structure,* which determines the definition of the *work packages,* i.e., the drawings, specifications, operations sheets, work sequences, and material lists defining what to do and how to do it.

The traditional approach to defining outfitting work packages is systems oriented, that is, the various ship systems (ventilation, electrical, communication, hydraulic, etc.) are considered separately. While this orientation follows naturally from the design phase, and simplifies the collection of production cost data by system, it is not the best orientation in terms of production scheduling and control. Only recently has a product oriented work breakdown structure (PWBS) been proposed [9] for U.S. shipyards. In essence, PWBS would yield work packages describing all outfit work required for a particular area in a hull block.

The second important aspect is the *structure of the scheduling process.* Figure 2 illustrates the relationships between major planning/scheduling documents and demonstrates both the hierarchical structure of the process and the bifurcation into separate steel and outfit schedules. The dashed lines in Figure 2 indicate that the System Schedule is developed subsequent to and is constrained by the Hull Block Erection Schedule and the more detailed Block Assembly Schedule. This dependence is just one result of the traditional treatment of outfitting as a successor function.

2. THE OUTFIT PLANNING PROBLEM

Current practice in planning and scheduling ship production inherently limits the ability to integrate steel and outfit activities. It results in the bulk of outfitting work being performed in the erected hull, either on the ways, after a block is closed in, or at the wet dock or outfit pier.

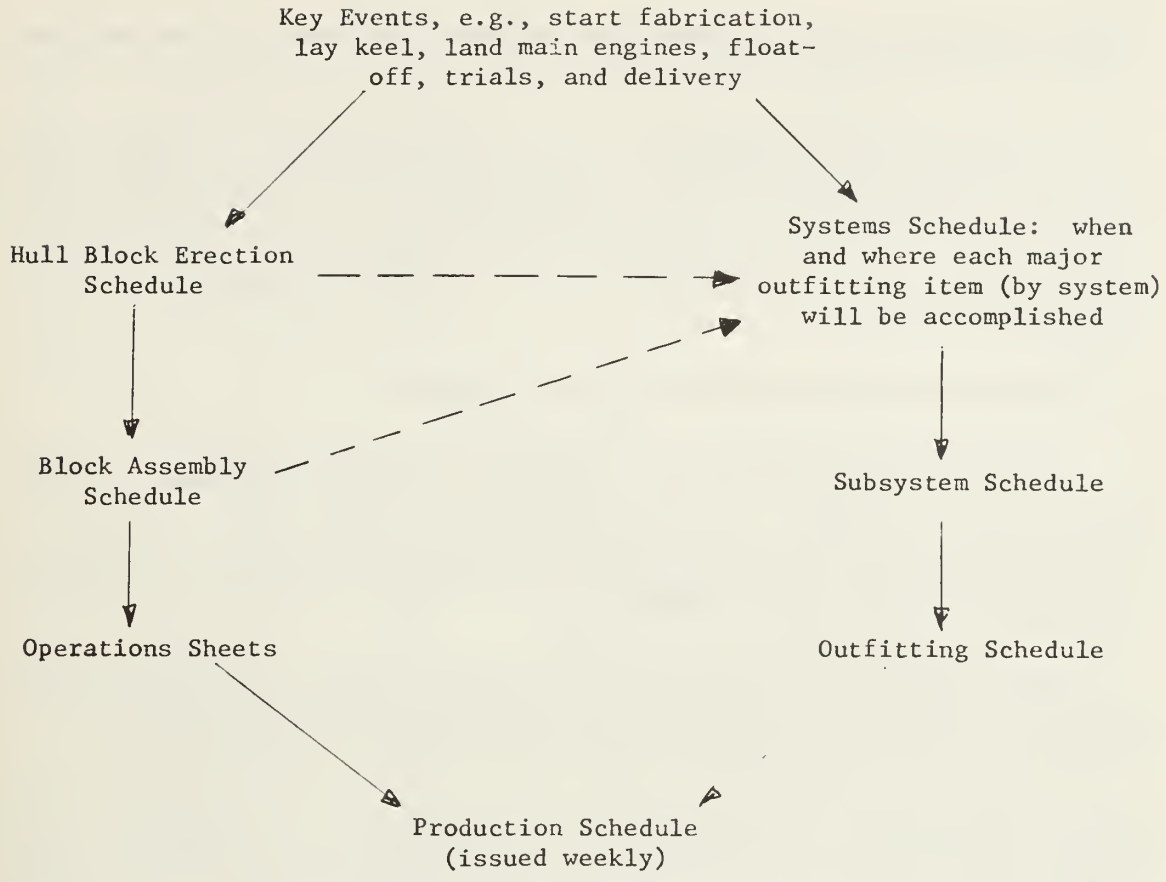
Key Events, e.g., start fabrication, lay keel, land main engines, float-off, trials, and delivery
Hull Block Erection Schedule
Systems Schedule: when and where each major outfitting item (by system) will be accomplished
Block Assembly Schedule
Subsystem Schedule
Operations Sheets
Outfitting Schedule
Production Schedule (issued weekly)

FIGURE 2. Structure of production scheduling

Working conditions in the hull are not ideal because of factors such as difficult access, limited space in which to work, difficulties in adequately venting noxious fumes, and difficult work positions (e.g., overhead welding). The workplace is typically congested, with high material flow costs, and often hazardous conditions.

It is now widely recognized that many of these problems can be relieved to some degree by doing more outfitting activities earlier in the production process, i.e., either in the assembly area or in the shop (vendor) area. To do this, however, requires a much greater integration of steel and outfit planning than has been the rule.

The fundamental problem is to identify economically desirable opportunities for preoutfitting. this requires answering two types of questions. The first is related to feasibility, i.e., "Is there sufficient time and resource available to do a particular outfitting activity in the assembly or shop area, and is it technically feasible?" The second question is one of economics "Is it more economical to preoutfit this activity?" What is needed is a systematic way to answer these questions.

Any such evaluation procedure must have two essential components. The first is a flexible work breakdown structure which will identify outfit activities with their geographical

location in the ship. The second component is a methodology for selecting the appropriate work breakdown and determining a feasible schedule.

The first component relates chiefly to work methods, while the second relates chiefly to scheduling. The required work breakdown structure has been used for many years in the most competitive foreign yards and is described in detail in [8] and [9]. It is summarized in the following paragraphs. The methodology for selecting a particular production plan and developing a feasible schedule has not yet been developed from a quantitative point of view. Section 3 presents a model of the associated decision problem, and Appendix A contains a mathematical statement of the model.

2.1 Product Oriented Work Breakdown and Zone Outfitting

The PWBS divides the shipbuilding process into three basic types of work, hull construction, outfitting, and painting, and further classifies each type of work as fabrication or assembly. Interim products are classified by resource requirements and certain product features such as type of *system* (e.g., lighting system) and *zone* (any geographical division of the ship). It is noted that PWBS bears a close resemblance to group technology. It is quite flexible and allows activities to be summarized in many different ways.

Zone outfitting is to outfit activities what hull block construction is to steel activities, i.e., it is a logical method for organizing the work to improve planning and productivity. Zone outfitting incorporates three *stages* for outfitting: *on-unit, on-block,* and *on-board.*

Outfitting on-unit refers to the assembly of an interim product consisting of only outfit materials. Examples are water distilling unit, fuel oil purifier unit, pipe passage unit, pump room flat unit, etc. Outfitting on-unit impacts the shop-related resources and the material handling facilities. It may require additional labor and materials for structural support to units to permit their movement to the assembly or ways areas. It also has some impact on hull contruction progress since the unit must be landed. However, "on-unit outfitting should be given the highest priority . . . because assembly is performed in shops which provide ideal climate, lighting, and access" [8].

Outfitting on-block refers to the installation of outfit components, or units, in a hull block in the assembly area prior to its erection on the ways. Outfitting on-block is more difficult than outfitting on-unit because it requires careful coordination between the steel activities and the outfit activities (recall that there are usually two distinct planning and scheduling functions) and may impact the duration of a block's occupation of an assembly area.

Outfitting on-board includes any required outfitting activity which has not been performed in either of the two previous stages. Although outfitting on-board describes outfitting as usually practiced, it also allows for nontraditional activities such as the connection of outfit units or outfitted blocks.

Figure 3 illustrates the possible material flows among the zone outfitting stages. Both on-unit and on-block outfitting stages correspond to the previously defined assembly mode of production, although either one could occur in a shop facility. Similarly, the on-board outfitting stage can correspond to either the erection mode or the outfitting mode as previously defined.

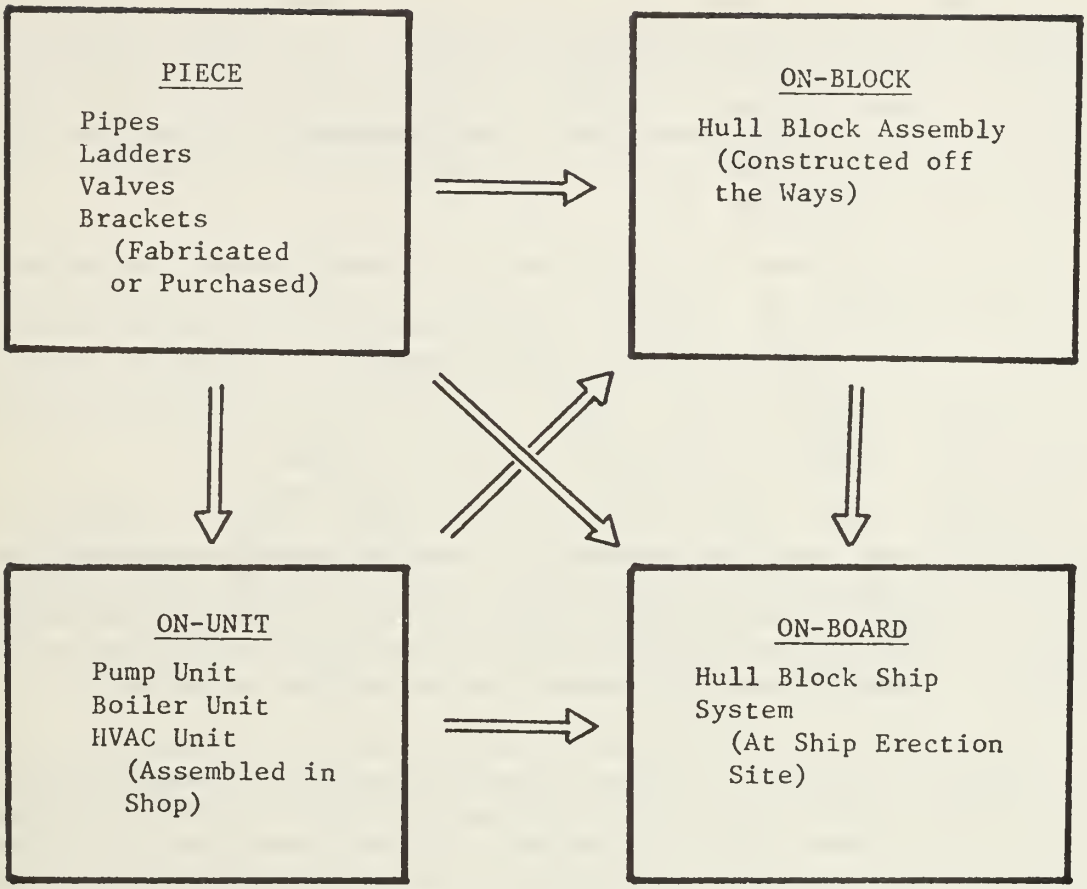

FIGURE 3. Outfitting stages

Clearly, a full implementation of the zone outfitting approach requires an outfit planner to think in terms of interim products, rather than systems. Furthermore, zone outfitting requires close coordination between steel and outfit schedules. Thus, zone outfitting and PWBS are mutually reinforcing approaches to ship production.

2.2 The Outfit Planning Problem

Because zone outfitting defines various stages for outfitting, it admits *alternatives* for the execution of outfit activities. Thus, the full exploitation of the zone outfitting concept requires that production management be able to resolve all the alternative choices available. The problem of choosing among the many available outfitting plans, which is referred to here as the outfit planning problem, is not currently faced by production management in U.S. shipyards, simply because options are not considered. The following paragraphs suggest the types of options that do exist.

Although there are options in zone outfitting, not every activity can be performed in all three of the outfitting stages. There are some outfit components which are only installed in the on-board stage e.g., furnishings and other similar materials which are subject to damage or

pilferage are always installed in the on-board mode. These will be designated *on-board components.* Of the remaining components, some are associated with distributed systems, e.g., wireways or ventilation ducting, rather than distinct units, e.g., pumps, motors, valves, etc. These will be referred to as *non-unit components*, since outfitting on-unit is not appropriate. Finally, there are the outfit components which could be identified by or associated with a specific unit. These will be referred to as *free components*, since any stage may be selected (although on-unit outfitting is preferred).

Note that these designations are fixed to some extent by design practices. For example, a given system consisting of, say, a pump and piping, may be conceived and designed in several ways. If it is treated simply as a collection of separate components which must be installed in the ship, then the components will have the "non-unit" designation. Alternatively, if the components are viewed as integral parts of a single unit or set of units, then they will have the "free" designation. Chirillo and Jonson [8] give examples of outfit components that may be associated with units, although this practice is not typical in U.S. shipyards.

Although a free outfit component can be associated with a specific unit, it need not be installed in the on-unit stage. The component may instead be installed on-block or even on-board. Non-unit components may be installed either on-block or on-board, but not, of course, on-unit. As indicated in the outfitting stage definitions, units may be installed either on-block or on-board. These relationships are summarized in Figure 4 where a three-way distinction is made between the component type, its production stage, and the production location.

Outfit planning requires, for each outfit component, a selection of outfit stage. The selection decisions are constrained by a number of factors. In particular, it is common practice to take the hull block erection schedule as fixed when planning the outfit activities. For example, each hull block has a fixed deadline for its completion, and at that point in time it is lifted onto the ways for erection. Thus, all on-block outfitting planned for that hull block must be completed before its erection date. Similarly, if a unit is to be installed in the block, all the associated on-unit outfitting must be completed in time to allow the unit to be moved onto the block and installed before the block erection date. Furthermore, if the block "closes in" any previously erected blocks, any large components (main engine, diesel generators, etc.) must be landed in these blocks prior to closing in.

The hull block erection schedule is a *constraint* in outfit planning because of convention. It is also possible to treat the hull block erection schedule as part of the decision process, i.e., if it were justifiable, a hull block might be delayed to allow more on-block outfitting to be performed. This practice does not appear to be in use currently in the U.S., and is not considered in the developments to follow. It is, however, common in Japanese shipyards, and may be adopted by U.S. yards in the future.

Another constraint which may affect outfit planning decisions in many yards is the available lifting capacity. Outfit units and outfitted hull blocks must not exceed the safe lifting capacity of the available equipment. Size is a similar consideration, i.e., units must be sized in light of the available access.

The effect of outfit planning decisions on limited yard resources must also be considered. Among the resources to be considered are labor and material availability and production or storage space. When determining outfit stages, care is required to insure that the resulting pro-

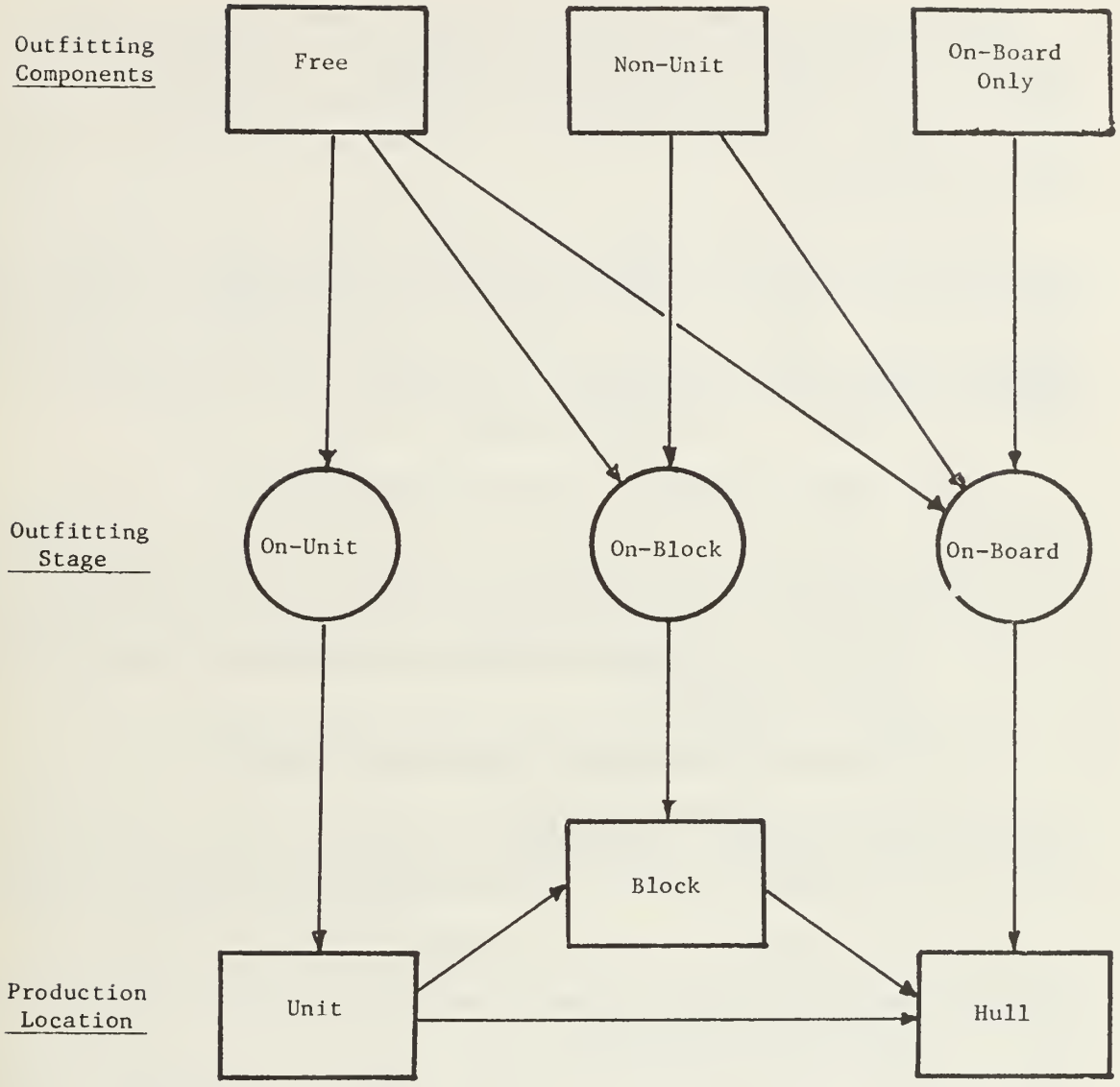

FIGURE 4. Relationships between outfit components, outfitting stages and production location

duction schedule does not call for more labor than is available in each affected craft and grade. Likewise, since production typically requires space and fabricated components or units may need to be stored temporarily, the available yard facilities must not be overcommitted.

These resource allocation considerations are perhaps the most difficult aspect of outfit planning, especially in situations where multiple ships are in production simultaneously. The reason is that in order to guarantee feasibility of the mode selections, a feasible *schedule* must be determined. The selection decisions and subsequent scheduling decisions interact in a complex fashion and cannot be made independently.

There has been no discussion as yet of the specific criteria by which the outfit plan is to be evaluated. Several criteria may be considered, all motivated by economic considerations. Considerable cost savings are indicated [6, 8] for outfitting on-unit and on-block, relative to outfitting on-board. These cost savings result from lower skill requirements, better material access, less congestion, better quality control, etc. One criterion, which should be minimized, is total cost of outfitting.

Another result of increased on-unit and on-block outfitting would be reduced delivery time. Reducing delivery time is favorable to both owner and builder, since the owner has use of his ship sooner and the builder receives final payment sooner. In addition, they both benefit from the reduced ". . . interest costs for the substantial accumulating investment represented by construction progress and for achieving maximum utilization of expensive facilities such as building dock" [8]. Thus a second criterion, to be minimized, would be completion time. In particular circumstances, yet other criteria might apply.

The *outfit planning problem* can now be stated more precisely as follows:

Given:

(1) a catalog of the outfit phase activities for which there are outfitting options;

(2) for each such activity, a list of the outfitting options, including time, resource and precedence requirements;

(3) the ship delivery schedule and any fixed milestone deadlines;

(4) labor availability by craft and grade;

(5) facility capacities and availabilities (lifting, dovered space, yard space, etc.); and

(6) other constraining factors (material availability, rate of cost accumulation, etc.).

Determine: The outfitting option to be used for each outfit phase activity considered, along with the necessary schedule.

The outfit planning problem is one of selecting from a number of interrelated options, a set of options that will satisfy the given resource constraints while optimizing some criterion, such as outfit costs or delivery date.

3. MODEL FORMULATION

Modeling the problem involves the use of activity network models such as CPM or PERT [29, 30, 31, 32, 33]. For practical as well as academic reasons (see, e.g., [14, 24]), only deterministic, i.e., CPM-like, models will be considered.

The use of *d*eterministic *a*ctivity *n*etworks, or DANs [16], to model ship production requires some assumptions about the ship building process.

A1: Ship specifications, such as production drawings, can be converted into well defined, distinct work packages, or activities.

A2: Assuming unlimited production resources, the only relationship between the activities is one of sequence or *precedence*. An activity, "A," precedes another activity, "B," if "A" must be completed before "B" can be initiated.

These two assumptions permit graphical representation of the relationship between production activities. The one used in this research is the *a*ctivity *o*n *n*ode, or AON, representation [16].

Note that assumption A2 does *not* limit the relationships between activities to precedence only. Other types of relationships are possible, for example, two activities may require the use of the same limited resource.

A3: Associated with each activity is information about its duration (including resource-time options), about its requirements for various resources, and about its due date (or completion deadline) if appropriate.

In order to use the DAN model in planning, it must include certain information about the activities or work packages beyond precedence relationships. At a minimum, each activity has a given duration and resource consumption. In addition, it is often the case that the activity duration depends on the *rate* at which resources are applied, i.e., there are resource-duration options. Start and due dates are often imposed because of special considerations beyond just the work content of the project, e.g., a hull erection schedule.

A4: The various resources required to perform the activities are explicitly defined and the availability of the resources over the planning horizon is specified.

The resources required by the activities can be of two types. Some resources are consumed as they are applied to production, e.g., steel which is applied to a particular hull block. Any subcontracted material falls into this category. This type of resource must be available when the associated activity is scheduled.

The other resource type is available at a certain *rate* rather than a total amount. For example, a given labor pool in a particular craft translates into a fixed number of man hours per day of that resource. Of course, over the long run, the number of man hours can be changed by changing the size of the labor pool. Thus, this type of resource is not "used up" in the same way that materials are.

Resources of this type present more difficult planning problems. One reason is that the cost of the resource depends on the rate at which it is used, i.e., if the resource is not fully utilized in some period, there is a wasted resource cost. Thus, one goal is to schedule the production activities so that resources of this type are always fully utilized.

The classical DAN models, such as CPM, are inadequate for the outfitting planning problem because they are based on the assumption of a single, unambiguous definition of the activities. In contrast, the essence of the outfit planning problem is to select a particular activity definition (i.e., select production modes) from among all the available alternatives. It will be necessary to extend the DAN models to incorporate this additional complexity and to develop the corresponding extensions to the analytic methodologies.

3.1 An Activity Network Model of the Outfit Planning Problem

The goal of the following discussion is the development of a conceptual model for the outfit planning problem which is consistent with an activity network based approach to planning and scheduling. It must be recognized at the outset that the process being modeled exists only hypothetically and that the model does not represent any existing process. It is apparently the case that, at the present time, very few U.S. shipyards employ activity network based planning or scheduling procedures in production, thus, the proposed model constitutes a significant departure from currently standard practice. On the other hand, it is also apparently true that interest in this type of methodology is growing in many U.S. shipyards, so that the proposed model is in line with longer term trends in the industry and is based upon some elements currently recognized in the industry.

3.1.1 Defining the Activities

Current practice in U.S. production planning ([1], appendix 4) calls for work packages of 200-2000 man-hours, involving a *single* craft or trade. For comparison, the Japanese practice [8] is to define work packages of 40-120 man-hours. The following developments are based on the premise that activity descriptions can be made at the level of the smallest fabricated component and then aggregated as necessary. Furthermore, an activity, as discussed in the previous section, may consume different resources, i.e., it may involve two or more crafts. The organizational and operational ramifications of this departure from standard practice will be explored later.

In developing the model, it will be useful to maintain the distinction between outfit *components*, which are associated with the outfit materials, and the outfit *activities*, which are associated with production, i.e., fabrication, assembly and installation. The outfit components were categorized as on-board, non-unit, or free, and outfit activities were classified as on-unit, on-block or on-board. The question which follows from this classification scheme is, "How are the activities corresponding to a given outfit component defined?"

A fundamental assumption about outfit planning is:

A5: On-unit outfitting is preferred to on-block outfitting, which is preferred to on-board outfitting.

This assumption implies that if there were no resource conflicts, or time constraints, outfitting would always be done as early as possible in the production process. It is the resource conflicts and milestone event deadlines which lead to deviations from this "ideal" outfitting plan.

Free Outfit Components

The free outfit components present the greatest latitude in planning production since they may lead to on-unit, or on-block, or on-board production activities, or to a combination. As a consequence, these are the activities that present the most difficulties in formulating the DAN model of outfit planning.

The "ideal" outfitting plan would call for maximum use of the on-unit stage, with resulting units being installed whenever possible. Thus, the following assumption is made:

A6: The outfit planning process creates for the free outfit components, a catalog of maximally outfitted units. For each unit, all the required materials, fabricated pieces and assembly work elements are specified. The set of outfit work elements for a given unit will be referred to as the *maximum outfit* set for the unit.

A particular unit from this catalog will generate many individual activities in the model (recall that the individual activities may be aggregated at a later step in the planning process). For example, each individual component of the unit must be either fabricated or purchased, resulting in the definition of either a fabrication activity or a purchasing activity. Each subassembly operation likewise results in the definition of a distinct activity.

An implicit requirement is that the units in this catalog are nonoverlapping, i.e., no free outfit component is a component of more than one unit. Thus, the definitions of the units themselves are considered as fixed at an earlier stage in the planning process. The problem of selecting from among alternative unit definitions is included in the proposed model in certain fairly restrictive situations as seen later.

It may be the case that selection of the on-unit outfitting rather than on-block or on-board "induces" additional work elements. For example, additional bracing may be required to prevent damaging the unit during handling and moving. Any such induced work must be reflected in additional activities in the DAN model.

Since the ideal outfitting plan may not be feasible given the available resources and milestone event deadlines, it is necessary to specify the alternatives to be allowed within the outfit planning model.

A7: For each unit, there is a set of outfit components which represents the least amount of on-unit outfitting that can be done and still be economically justifiable. The associated set of outfit work elements will be called the *minimum outfitting* set for the unit.

This assumption implies that if a particular unit is selected from the catalog for on-unit outfitting, it need not be completely outfitted. However, it will include at least those outfit work elements contained in its minimum outfitting set. Associated with the maximum and minimum outfit sets are related sets of outfit components, designated as the maximum and minimum outfit *kits*.

A given unit may be assembled in the on-unit mode. If so, it must include all components in its minimum outfit kit and it may include any additional components in its maximum outfit kit. Any work elements from the maximum outfit set which are not selected for on-unit outfitting must be performed at a subsequent stage, i.e., either on-block or on-board. The outfitted unit itself also may be installed on-block or on-board. If installed on-block, its assembly and installation must be completed before the block erection deadline.

These possibilities can be incorporated in a CPM-like precedence diagram as illustrated in Figure 5. In this example, nodes 1-4 represent the purchase, fabrication, or subcontracting activities for outfit components in the maximum outfitting kit for some unit. The components corresponding to nodes 2, 3 and 4 are in the minimum outfitting kit for the unit, i.e., at least these components must be included if the unit is selected for fabrication.

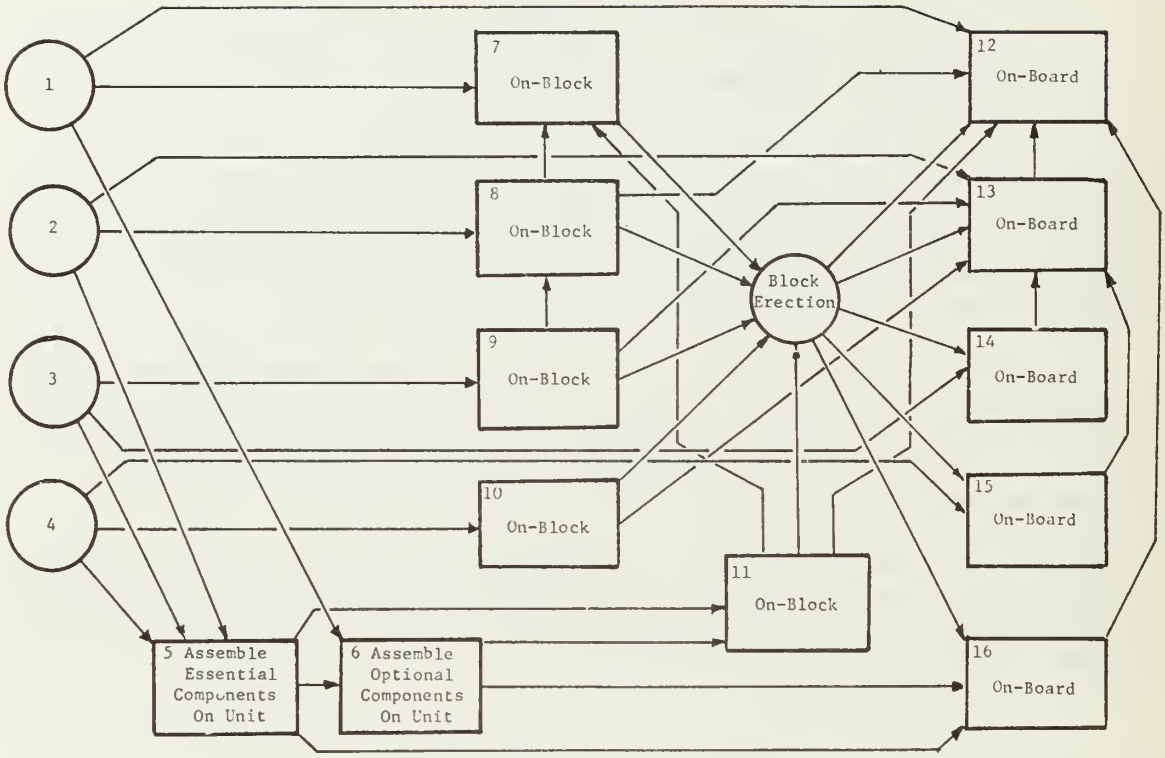

FIGURE 5. Activity model for free outfit components

A component, such as the one corresponding to node 1 in the figure, which is in the maximum outfitting kit has the following characteristic. It is a component which could be included in the unit fabrication and, in fact, it would be desirable to include it. However, if there are frustrating circumstances, for example, insufficient fabrication lead time or insufficient resources (labor, equipment, or material), then such a component may be left off the unit. It is, in a sense, an auxiliary component of the unit. On the other hand, components in the minimum outfitting kit are considered essential to the unit, so much so that they cannot be omitted from the unit.

The activities represented in the diagram by square nodes are the ones subject to the outfit planning decisions, which designate the specific stage of outfitting for each component.

To insure that components produced by the first four activities in Figure 5 are actually installed, the outfit planning decisions must obey the following guidelines:

(1) *Exactly* one activity is selected from each of the sets:
{6,7,12} to insure that component 1 is included;
{5,8,13} to insure that component 2 is included;
{5,9,14} to insure that component 3 is included;
{5,10,15} to insure that component 4 is included;
If activity 5 is selected, activities 8, 9, 10, 13, 14, and 15 cannot be selected.

(2) Activity 6 can be selected only if activity 5 is selected;

(3) If activity 5 is selected, then either 11 or 16 must be selected; if 5 is not selected, neither 11 and 16 can be.

If these three guidelines are followed, then a feasible solution will be constructed for the outfit planning problem. Note that if an activity is not selected, it simply becomes a discarded option, i.e., it does not affect subsequent scheduling or resource allocation decisions.

The example illustrates additional details that can be incorporated in this type of model. For example, if in addition to the minimum outfitting kit, component 1 is also to be included in the on-unit outfitting (i.e., activity 6 is selected) the associated work element, activity 6, must be completed by the time the unit is installed, either on-block (activity 11) or on-board (activity 16). This is indicated by the precedence relationships (5, 6), (6, 11) and (6, 16).

In this example, there is a required sequence for installing the outfit components; component 2 cannot be installed until after components 3 and 4 have been installed, and component 1 cannot be installed until after component 2 has been installed. These restrictions are satisfied by requiring that activity 13 has as its predecessors, either 9 or 14, and either 10 or 15. Similarly, activity 12 has as predecessors either 8 or 13. Note also that if activity 5 is selected (i.e., the unit is assembled) then on-block or on-board outfitting for component 1 must *follow* installation of the unit.

Finally, note that the block erection schedule can be introduced into the model simply by specifying a due date for the unnumbered node corresponding to block erection. One additional consideration was left out to simplify the figure and the discussion. It might be desirable to treat on-board outfitting as two distinct stages, one corresponding to prefloat off outfitting and one corresponding to wet-dock outfitting. This consideration could be affected within the model simply by defining four additional nodes, one for each of the four outfit components, and adding the necessary precedence relationships. This is illustrated for the previous example in Figure 6.

Non-Unit Outfit Components

The non-unit outfit components involve fewer production options than the free outfit components and it is therefore considerably easier to define the alternative activities generated by them. In fact, non-unit components generate a subset of the activities generated by free components. For instance referring to the example of Figure 5, suppose the on-unit outfitting activities, which are activities 5, 6, 11, and 16, are omitted. The resulting activity network would describe the options available for non-unit components 1-4.

In addition to sequencing requirements among the non-unit components, there may also be sequencing requirements between the non-unit components and certain free components or their associated units. The various types of relationships are summarized in Figure 7. As indicated in the figure, the model must account for the possibility of sequencing requirements between the non-unit components and certain free components or their associated units, as well as between the non-unit component and certain on-board components.

As with the free outfit components, it is conceptually easy to extend the model to allow two distinct on-board outfit stages. The illustration will not be repeated.

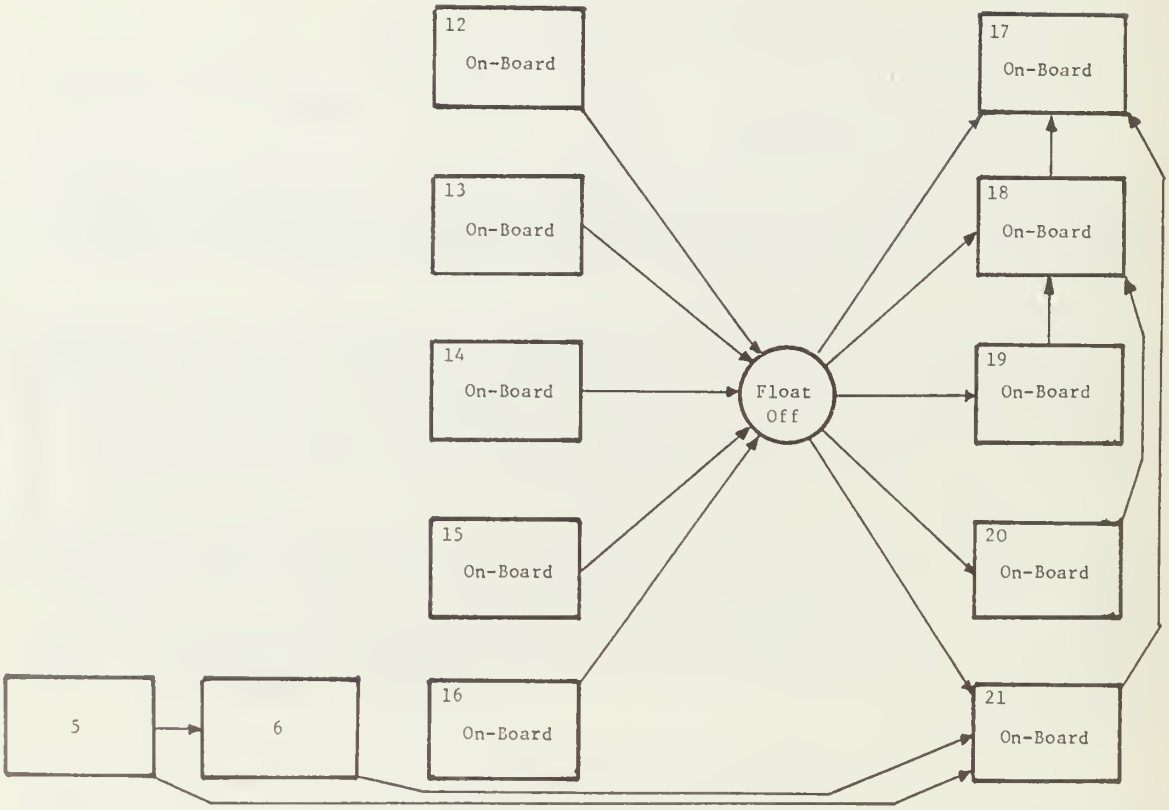

FIGURE 6. Adding two substages to the model

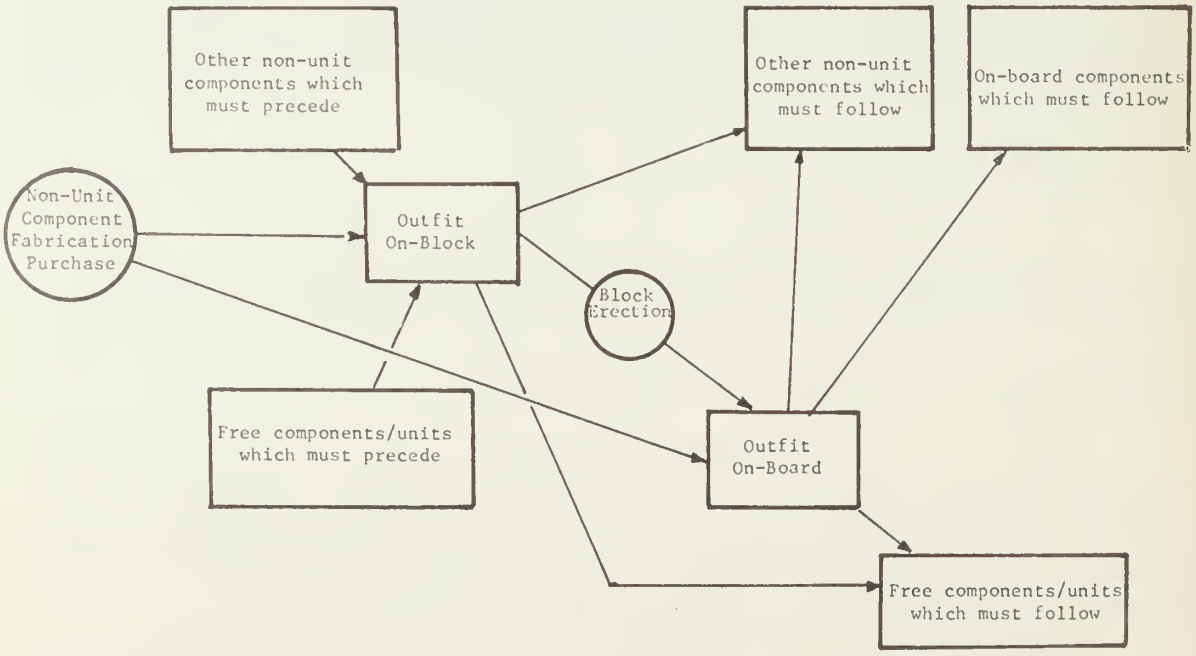
Other non-unit components which must precede
Non-Unit Component Fabrication Purchase
Outfit On-Block
Block Erection
Free components/units which must precede
Other non-unit components which must follow
On-board components which must follow
Outfit On-Board
Free components/units which must follow

FIGURE 7. Activity model for non-unit components

On-Board Outfit Components

The on-board outfit components require no outfitting mode decision, unless the possibility of two on-board stages (pre-float off and wet-dock) are allowed. In this case, each component generates two alternative outfitting activities with precedence relationships as shown in Figure 8. The requirement, then, is to select exactly one of the two activities.

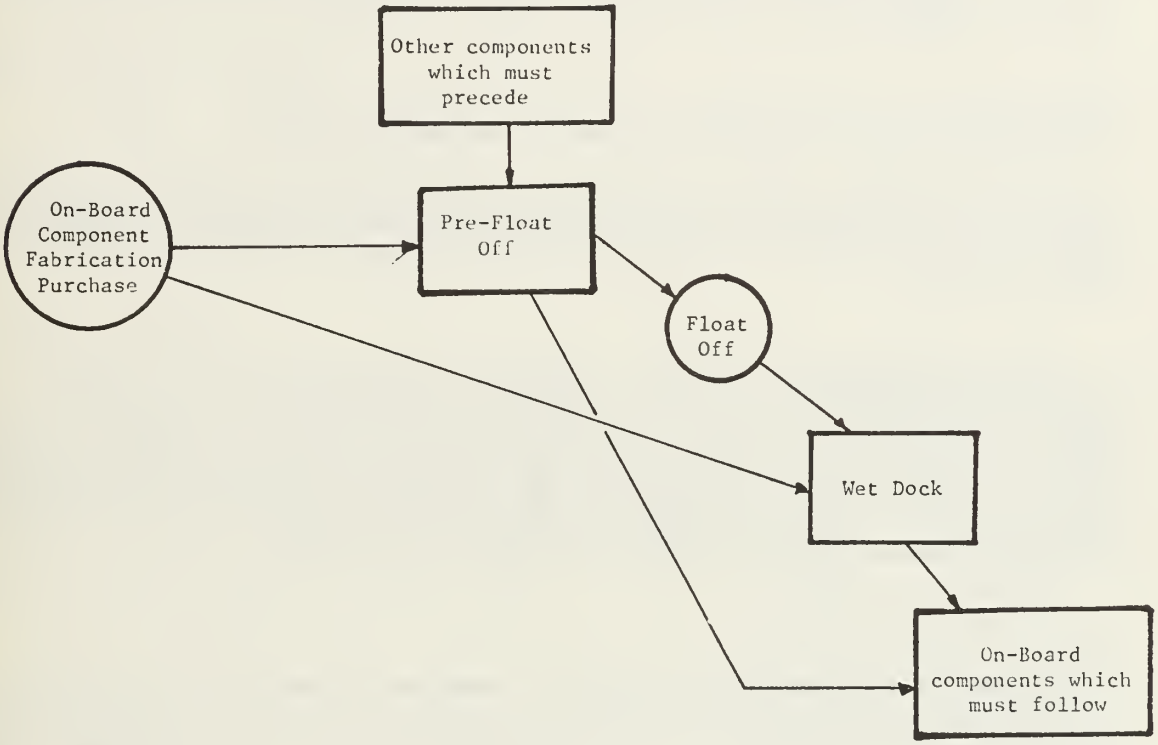

FIGURE 8. Activity model for on-board components

3.1.2 Defining the Decisions and Constraints

The fundamental decisions required in the outfit planning problem is the resolution of the options associated with each outfit component. This *selection decision* considers the activity network and requires a choice of exactly one of the alternative outfit activities for each outfit component (and perhaps the resulting unit). The selection decision must satisfy the sequencing requirements which are represented in the activity network as arrows. The sequencing requirements are *constraints* on the selection decision.

If there were no other constraints, the selection decision would be trivial because of assumption A5, i.e., each component would be outfitted as early as possible in the production process. There are, however three major types of constraints which may be violated by such a selection:

(1) [Time] The sequencing requirements may lead to a longer production time at some stage than is available from the given block erection and float-off milestones.

(2) [Labor] Even if there is sufficient time, the activities selected to be performed between two milestones may require more labor hours than are available in the crafts.

(3) [Weight and Size] Even if there is sufficient time and labor, the number of components selected for a unit or the number of units and components selected for a block may lead to a unit or block which is too large for the available facilities or access.

The constraints on time and weight may be easily checked once the selection decision is known. Such is not always the case, however, for the labor availability constraints.

In order to know whether or not a labor availability constraint is satisfied, a *schedule* for the activities must be specified. Thus, in situations where labor availability is a limiting factor, solving the outfit planning problem requires making a *scheduling decision* in addition to the selection decision.

The scheduling decision by itself is an extremely complex one. In fact, given the selection decision, the problem to be solved in making the scheduling decision is a "resource constrained CPM problem," [5, 12, 49, 50]. At the present time there is no optimization algorithm capable of solving large instances of this type of problem (see Bennington and McGinnis [5]) and based on recent results in combinatorics ([28], [44]), there is little hope that such an algorithm is possible. Thus, if solving the outfit planning problem requires a specific scheduling decision, any practical solution methodology will be heuristic in nature.

3.1.2.3 Defining the Criteria

The final step in formulating a model of the outfit planning problem is to define the criteria by which solutions are to be evaluated. The problem of evaluation is complicated by the fact that there are two distinct kinds of decisions being made: outfitting stage selection and activity scheduling. Furthermore, a number of different viewpoints could be considered, each leading, possibly, to a different criterion.

The viewpoint adopted here is that the outfit planning problem is to be solved in the context of a number of prior, exogenous decisions which fix many of the outfit planning problem parameters. For example, the milestone event times (such as lay keel, float off, delivery, etc.) are assumed fixed, along with the detailed block fabrication and erection schedule. (Note, however, that this analytic framework could be used in deciding on the appropriate milestone schedule.) Resource availabilities are considered as exogenous factors.

Within the environment resulting from these exogenous factors, the goal in outfit planning is narrowed to that of minimizing the cost of outfitting. Conceptually, then, all that is required is to estimate the outfitting cost associated with each of the outfitting alternatives. The best outfitting plan is the one with the smallest total cost. The goal of the scheduling component of outfit planning is to maximize labor utilization. This is accomplished when there are no periods in which the scheduled work content is less than the available labor.

While these two criteria are conceptually simple, their application may be difficult. In the first place, they require a significant effort in detailed estimation. The labor content, material and overhead costs, and duration must be estimated for each of several alternative outfit methods for a large number of outfit components. Current practice may not require such a detailed estimate for even one alternative. Clearly, procedures and methods will need to developed for aggregating outfit components in the activity network and for semi-automating the estimation at the necessary level of detail. The information required for this estimation process will have to be accumulated over time as there is more experience with on-unit, on-block, and on-board outfitting.

A preliminary and crude approach to the first criterion is the following. Assume that the savings to be realized by outfitting earlier in the production process is a constant fraction of the total cost to outfit on-board. The fraction could vary with the type of outfitting (e.g., electrical vs. hydraulic systems), or with the total cost of the outfitting activity or some other factor. The criterion then becomes one of maximizing the total savings over outfitting completely on-board.

3.2 Model Evaluation

A mathematical model has been developed to describe the outfit planning problem and appears in Appendix A. This model is in the form of a mixed integer programming problem and, consequently, it presents formidable difficulties in solution. In fact, recent theoretical development [28] have been interpreted as indicating that such problems (referred to as "NP-complete") *cannot* be solved optimally. Certainly it is true that, currently, practical problems of this ilk are not optimized. There are, however, a number of heuristic solution procedures which have been developed and used successfully to solve similar problems (e.g., see [38]).

Obviously, the model by itself cannot lead to better outfit planning. What is required is a systematic implementation of the model. There are several requirements for a successful implementation of the model, and these can be more easily discussed by referencing the diagram of Figure 9.

One of the requirements for a successful implementation is an appropriate methodology for solving the selection and scheduling problem for given milestone events and resource availabilities. As was indicated earlier, there is little hope for a general optimizing method for solving this problem, so in the most general case, the solution procedure will be heuristic.

The model requires large amounts of information and generates large numbers of detailed decisions. Thus, any practical implementation will require a fairly detailed, production oriented data base to support the solution procedure. Although many shipyards do not have such a data base at the current time, the SPARDIS system used by NASSCO [45] is one example of the type of system that would be required.

A third requirement is that the outfit planning process could in fact provide all the information required in the model. It appears that a major shift from current practice would be the idea of allowing (and therefore planning for) several alternative ways to accomplish the outfitting tasks. In addition, the use of the on-unit, on-block, on-board approach to outfitting is not currently widespread, although it is being strongly supported as a means for improving productivity [8].

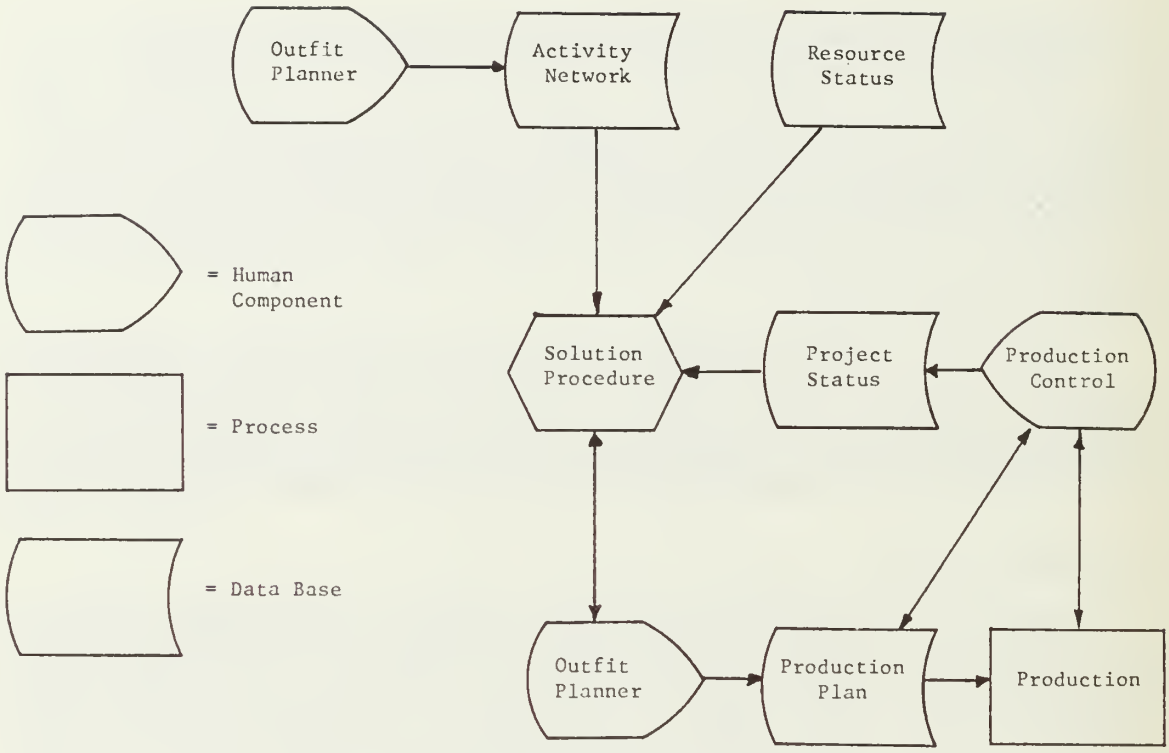

FIGURE 9. Outfit planning process

Given that the zone outfitting approach has been adopted, defining the alternative outfit activities discussed earlier should be straight-forward, albeit somewhat time-consuming. Observe that to a large degree, the outfit elements are associated with particular blocks. Therefore, an activity network resembles a large number of small subnetworks (one for each block) which are loosely connected by milestone events. It will be possible to "decompose" the network definition into smaller, more manageable tasks.

Figure 9 also indicates how the model might be used in practice. The use of the model for planning the outfitting of a ship as self-evident. Probably as important is the use of the model to "replan" when there are major deviations from the original plan, e.g., due to weather, change orders, priority repair work, etc.

A final point of discussion is the benefit to be obtained by the use of the model. The foremost benefit of the model, per se, is tighter planning and control of outfitting, resulting in higher productivity (and thus lower costs). In project-type work, such as ship production, it is important to correctly estimate the labor content of the work and then plan the work so that labor resource utilization is maximized. The proposed model provides a systematic means for coping with and coordinating the vast number of relationships which simply cannot be handled by an unaided human planner.

A secondary benefit from the proposed model is that it complements and strengthens the implementation of the on-unit, on-block, on-board approach to outfitting. It provides a systematic framework for identifying opportunities for on-unit and on-block outfitting as well as for determining the technical and economic feasibility of outfitting plans.

APPENDIX A: THE MATHEMATICAL MODEL

In developing the conceptual model, two types of decisions were identified: selection decisions and scheduling decisions. It will be convenient to formalize the selection decisions first. Associate with each outfit component an index, i, where $i = 1, 2, \ldots, N$, N being the total number of outfit components. Similarly, associate with each outfit unit an index $j = 1, \ldots, M$, and with each block an index $b = 1, \ldots, B$.

The selection decisions will be represented by indicator variables. For a particular component, i, the variables are:

$$x_i^u = \begin{cases} 1 \text{ if component } i \text{ is outfit on-}u\text{nit} \\ 0 \text{ otherwise} \end{cases}$$

$$x_i^b = \begin{cases} 1 \text{ if component } i \text{ is outfit on}-b\text{lock} \\ 0 \text{ otherwise} \end{cases}$$

$$x_i^h = \begin{cases} 1 \text{ if component } i \text{ is outfit on-board} \\ (\text{in the } h\text{ull}) \\ 0 \text{ otherwise} \end{cases}$$

Exactly one of the indicator variables must equal one for any component. However, not all stages can be selected for each element. Therefore, group the indices as follows:

$F =$ set of indices of free outfit components

$N =$ set of indices of nonunit outfit components

$B =$ set of indices of on-board outfit components

These sets are pairwise disjoint. Now the component selection decisions must satisfy:

(1) $$x_i^u + x_i^b + x_i^h = 1 \qquad i \in F$$

(2) $$x_i^b + x_i^h = 1 \qquad i \in N$$

(3) $$x_i^h = 1 \qquad i \in B.$$

Note that there is only one on-board option. The model can be readily extended to allow for pre-float off and wet-dock on-board outfitting. In order to simplify the exposition, this extension is not included.

There are similar indicator variables associated with each unit:

$$z_j = \begin{cases} 1 \text{ if unit } j \text{ is selected for assembly} \\ 0 \text{ otherwise} \end{cases}$$

$$y_j^b = \begin{cases} 1 \text{ if unit } j \text{ is installed on-block} \\ 0 \text{ otherwise} \end{cases}$$

$$y_j^h = \begin{cases} 1 \text{ if unit } j \text{ is installed on-board} \\ 0 \text{ otherwise} \end{cases}$$

Since a unit cannot be installed unless it is first assembled, the unit selection variables must satisfy the following constraint:

$$y_j^b + y_j^h - z_j = 0 \quad \forall \text{ j.} \tag{4}$$

The unit selection decisions and element selection decisions must be tied together. Define the following index sets:

$L(j)$ = set of indices of components in the minimum outfitting kit for unit j

$M(j)$ = set of indices of components in the maximum outfitting kit for unit j

The element and unit selection variables must satisfy:

$$\sum_{i \in L(j)} x_i^u - ||L(j)||z_j = 0 \quad \forall \text{ j} \tag{5}$$

$$\sum_{i \in M(j)} x_i^u - ||M(j)||z_j \leqslant 0 \quad \forall \text{ j} \tag{6}$$

where $||S||$ is the number of elements of the set S.

Constraint (5) requires that if unit j is selected ($z_j = 1$), then all the components in the minimum outfitting kit for that unit also must be selected. Constraint (6) permits additional components to be included in the unit only if the unit is fabricated.

The constraints (1)-(6) are *logical constraints* and merely guarantee consistency between the indicator variables and the decisions they represent. In addition, there are *structural constraints* which must be satisfied. One of these is the precedence relationships defined by sequencing requirements. Define

$P(j)$ = index set of components (units) which must precede component (unit) j in production.

Then the precedence constraints on the selection decisions are:

$$x_i^u + 2x_i^b + 3x_i^h - (x_j^u + 2x_j^b + 3x_j^h) \leqslant 0 \quad i \in P(j) \tag{7}$$

$$y_j^b + 2y_j^h - (y_k^b + 2y_k^h) \leqslant 0 \quad j \in P(k) \tag{8}$$

Constraints (7) and (8) require that for any component or unit, its predecessors must be outfitted or installed at the same or an earlier production stage.

A second category of structural constraints limits the total weight added to a unit or block. Note that these limits may be facility dependent, i.e., units fabricated in different shops may have different weight limits.

$$\sum_{i \in M(j)} w_i x_i^u \leqslant W_j \quad \forall \text{ j} \tag{9}$$

$$\sum_{j \in U(b)} \left(\sum_{i \in M(j)} w_i x_i^u \right) y_j^b + \sum_{i \in F(b) \cup N(b)} w_i x_i^b \leqslant W_b \quad \forall \text{ b} \tag{10}$$

where

w_i = weight added by outfit element i

W_j = maximum weight allowed for unit j

$U(b)$ = units which go into block b

$N(b)$ = subset of components of N which go into block b

$F(b)$ = subset of componnts of F which go into block b

W_b = maximum outfitting weight added to block b.

The first term in constraint (10) is the total weight of units which are selected for installation on-block. The second term is the total weight of components (not part of a unit) which are outfitted on-block.

In order to deal with the time and labor availability constraints, the scheduling decisions must be formalized. Define the following scheduling variables:

t_i = scheduled start time for component i outfitting

θ_j = scheduled time for completing unit j fabrication

τ_j = scheduled start time for unit j installation .

The scheduling variables must satisfy all the precedence constraints as well as the scheduling limitations imposed by the steel schedule.

First, consider the constraints involving on-unit outfitting.

$$t_i x_i^u + d_i^u x_i^u - t_j x_j^u \leqslant 0 \quad i \in P(j) \tag{11}$$

where d_i^u = time to outfit component i on unit. Constraint (11) requires that all predecessors of component j must be completed before component j can be outfitted on unit.

$$t_i x_i^u + d_i^u x_i^u - \theta_j z_j \leqslant 0 \quad \forall \text{ i } \in \text{M (j)} \tag{12}$$

Constraint (12) requires all on-unit outfitting to be completed before the unit itself is completed.

$$\theta_j + d_j - \tau_j \leqslant 0 \quad \forall \text{ j} \tag{13}$$

where d_j = material handling delay for unit j.

Constraint (13) is included to allow for possible significant material handling delay or resource requirement.

The installation of units and outfit components on-block must not only satisfy precedence but "schedule window" constraints as well.

(14) $$\tau_j y_i^b + d_j^b y_j^b - \tau_k y_k^b \leqslant 0 \quad j \in P(k)$$

where d_j^b is the time required to install unit j on-block.

Constraint (14) forces the on-block installation of unit k to be after the on-block installation of its predecessors.

(15) $$\tau_j - T_b^s \geqslant 0 \quad \forall\, \mathrm{j} \in \mathrm{U(b)}$$

where T_b^s = earliest possible time for on-block outfitting on block b.

Constraint (15) forces the installation of the unit j to be after the time when installation is feasible.

(16) $$\tau_j y_j^b + d_j^b y_j^b - T_b^f \leqslant 0 \quad \forall\, \mathrm{j} \in \mathrm{U(b)}$$

where T_b^f = latest possible time to complete outfitting on block b.

Constraint (16) sets the deadline for on-block installation of units.

There are similar precedence and schedule window constraints for the on-block outfitting of free and nonunit components:

(17) $$t_i x_i^b + d_i^b x_i^b - t_j x_j^b \leqslant 0 \quad j \in P(i)$$

(18) $$t_i x_i^b - T_b^s\, x_i^b \geqslant 0 \quad i \in F(b) \cup N(b), \quad \forall\, \mathrm{b}$$

(19) $$t_i x_i^b + d_i^b x_i^b - T_b^f \leqslant 0 \quad i \in F(b) \cup N(b), \quad \forall\, \mathrm{b}.$$

These same precedence and schedule window constraints are repeated for both units and elements for on-board outfitting. For the units, the constraints are:

(20) $$\tau_j y_j^h + d_j^h y_j^h - \tau_k y_k^h \leqslant 0 \quad j \in P(k)$$

(21) $$\tau_j - T_h^s y_j^h \geqslant 0 \quad \forall\, \mathrm{j}$$

where T_h^s = earliest possible time for installing unit on-board.

(22) $$\tau_j y_j^h + d_j^h y_j^h - T_h^f \leqslant 0 \quad \forall\, \mathrm{j}$$

where T_h^f = latest possible time for installing unit on-board.

For the outfit components, the corresponding constraints are:

(23) $$t_i x_i^h + d_i^h x_i^h - t_j x_j^h \leqslant 0 \quad j \in P(i)$$

(24) $$t_i x_i^h - T_h^s x_i^h \geqslant 0 \quad \forall\, \mathrm{i}$$

(25) $$t_i x_i^h + d_i^h x_i^h - T_h^f \leqslant 0 \quad \forall\, \mathrm{i}.$$

In addition to precedence and schedule window constraints, the scheduling decisions must be feasible with regard to the resource availabilities. Resource availability constraints are quite difficult to formulate in explicit terms, so the following approach is typically used (see, e.g., models in [5] and [11]). Define the following:

$A_e(t)$ =set of outfit components being outfitted at time t
$A_u(t)$ =set of outfit units being installed at time t
r_{icu} = level of resource category c required by component i when outfitted on-unit
r_{icb} = level of resource category c required by component i when outfitted on-block
r_{ich} = level of resource category c required by component i when outfitted on-board
r_{jcf} = level of resource category c required to fabricate unit j
r_{jcb} = level of resource category c required to install unit j on-block
r_{jch} = level of resource category c required to install unit j on-board
R_{ct} = level of resource category c available at time t.

Now the resource availability constraints are:

$$\sum_{i \in A_e(t)} (r_{icu}x_i^u + r_{icb}x_i^b + r_{ich}x_i^h) +$$

$$\sum_{j \in A_u(t)} (r_{jcf}z_j + r_{jch}y_j^b + r_{jch}y_j^h) \leqslant R_{ct}. \tag{26}$$

The difficulty with using such a constraint is that the sets $A_e(t)$ and $A_u(t)$ depend on the scheduling decisions. In fact this is, to a certain degree, the nub of the resource constrained project scheduling problem.

The constraints (1)-(26) can be shown to be redundant. For example, if the scheduling related precedence constraints, (11)-(25), are satisfied, then the selection related precedence constraints, (7) and (8), must necessarily be satisfied. The reason for including the redundant constraints, (7) and (8), is to allow for solution procedures which try to decouple the selection and scheduling decisions.

Since the criterion specified for the outfit planning problem is to minimize outfitting costs, define:

C_{iu} = cost of outfitting element i on-unit

C_{ib} = cost of outfitting element i on-block

C_{ih} = cost of outfitting element i on-block

C_{jb} = cost to install unit j on-block

C_{jh} = cost to install unit j on-board.

The objective function for the mathematical model is:

$$\text{Minimize } \sum_i [C_{iu}x_i^u + C_{ib}x_i^b + C_{ih}x_i^h] + \sum_j [C_{jb}y_j^b + C_{jh}y_j^h].$$

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RECRUITERS, ADVERTISING, AND NAVY ENLISTMENTS

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ABSTRACT

Regression analysis is used to estimate the effects on Navy enlistments of recruiters, advertising, unemployment and other factors. In measuring the effects of these factors on enlistments, changes in both the demand for and supply of enlistments are taken into account. Advertising is treated as a capital investment; its effect is estimated using a maximum likelihood technique that was developed for measuring the effects of capital investments. We find strong evidence that recruiters and unemployment increase enlistments. Advertising also seems to increase enlistments, but its effect is highly uncertain. The results suggest that increases in recruiting resources would have eliminated the enlistment shortfalls experienced by the Navy in FY 1978-79.

INTRODUCTION

Navy recruiting efforts were largely successful unitl FY 1978. Since the advent of the All-Volunteer Force (AVF) in 1973, the Navy frequently achieved its enlistment goals or only slightly fell short. However, shortfalls averaged 10 percent in FY 1978 and FY 1979, and they are expected to increase further in the 1980s because of a decline in the youth population.

Similar recruiting problems have also been experienced by the other Services. As a result, some view the AVF as a failure and urge that we return to the Draft. But a draft involves many hidden social costs and inequities which make it an undesirable option.* Instead, solutions consistent with an AVF are likely to be less costly to society, if not the Defense Department's manpower budget, and such solutions should be chosen.

The Services' major difficulty has been an inability to recruit enough male *high school graduate* enlistments (HSGs). One set of AVF solutions involves options that would reduce demand for them. This could be accomplished, for example, by substituting for male HSGs, nonHSGs and HSGs in the lower mental groups as well as women and civilians.

Another set of options would increase the supply of HSGs. One frequently recommended by economists is to raise military pay and benefits, but this is a costly option: higher pay must be given to the entire stock of manpower as well as to new recruits; enlistment bonuses are a more selective instrument, but they too must be paid to a large population (all enlistees).

Another option is to increase the amount of resources devoted to recruiting, i.e., *recruiters* (R) and *advertising* (A). This approach may be less expensive than raising pay or giving

*For discussion see Reference [6], Appendix M.

bonuses because it eliminates extra payments or rents to those who would volunteer anyway. Unfortunately, there is little evidence on the productivity of R and A, so it is difficult to estimate the cost of using more to reduce shortfalls. To help evaluate such an alternative, this study analyzes the effects of recruiters and advertising on Navy HSGs.

Previous Studies

There have been numerous studies on enlistments.* In these regression analysis was used to relate the enlistment rate, e.g., HSGs per capita, to factors such as pay and unemployment. Typically, it is assumed that HSGs are limited solely by supply factors and, therefore, the flow of HSGs is independent of demand considerations. But this is a questionable assumption because the demand for enlistments seems to have had a strong effect on HSGs. As illustrated in Figure 1, there is a strong relationship over the period FY 1972-78 between HSGs and the demand for total enlistments, i.e., quotas. The relationship seems to hold for all HSGs as well as those in the upper mental groups (MG1-3U).† HSGs seem to be strongly affected by the demand for manpower.‡ Thus, we observe a reduced form equation for HSGs rather than a supply function as assumed by previous researchers.

What might cause a positive association between quotas and HSGs? Perhaps recruiters simply worked less hard when faced with low quotas. Such a response seems reasonable as long as recruiting goals are achieved, which was largely the case prior to FY 1978.

In FY 1978, however, there were large shortfalls; as a result we presume a high level of recruiter effort — yet there were relatively few HSGs. Thus, low recruiter effort does not explain why there were few HSGs in FY 1978. Still, changes in recruiter effort might partially explain Figure 1 in earlier years.

A relationship between quotas and standards may better explain Figure 1 as well as the FY 1978 results. The Navy has tended to lower standards to reduce shortfalls and raise them when faced with an excess supply of applicants.§ For example, when faced with high quotas and the prospect of large shortfalls in FY 1972, standards were lowered enabling more high school graduates in the lower mental groups to enlist in the Navy.

As quotas changed, changes were brought about in certain supply factors. Whatever they are, recruiters' efforts or standards, it is necessary to control for their effects to measure the productivity of R and A.**

*For examples, see Reference [1-3], [5-15].

†There are five mental groups. The upper mental groups are one, two, and the upper half of three.

‡The Navy's demand for manpower is essentially independent of the supply of HSGs. Quotas are based on projections of total manpower requirements rather than on projections of supply factors such as unemployment rates.

§For evidence on the relationship between quotas and standards, see Reference [15], pp. 4-5, 17 and 28-29.

**During FY 1972-77, the Navy permitted standards to fall as a means of meeting enlistments goals. We suspect that what has happened recently, is that the Navy has set high standards and has decided to hold the line on them. Moreover, standards are probably so high that any past slack in recruiter effort has now largely disappeared. Because changing standards and recruiter effort no longer seem to be available as equilibration mechanisms, the Navy must look to other supply variables as a means of reaching its enlistment goals, or it must lower those goals by substituting other cohorts for male HSGs.

The study asks how many HSGs would be generated by additional R and A. We answer this question using regression analysis of quarterly data for 3Q 1971 - 4Q 1977.* Our methodology is unusual in adjusting for the effects of changes in demand and in the treatment of advertising which is considered a capital investment. The study provides the first estimate of the productivity of Navy advertising in generating enlistments.

REGRESSION MODEL

Supply Factors

We assume the supply of HSGs to the Navy depends on the following factors: (1) HSG recruiting efforts; (2) enlistment standards; (3) awareness of the Navy as a career choice; (4) relative economic benefits of enlisting; and (5) size of the youth population. Measures of these factors will be used to analyze high school graduate enlistments.

HSG Recruiting Efforts

Recruiters have more than a passive role; they visit high schools, for instance, and give talks on the advantages of a Navy career. We suspect that differences in recruiters' efforts have had measurable effects on HSGs. We therefore choose as a factor the effort recruiters made to enlist high school graduates.

This factor requires special handling because it cannot be measured directly: The Navy collects no systematic data on how much time recruiters spend in active search of high school graduates. We will have to use a number of variables to measure the effects of this factor.

The desired measure—recruiter man-years devoted to enlisting high school graduates—depends on the size of the recruiting force, their overall effort, and the percentage of time devoted to recruiting high school graduates. The total number of personnel with a recruiter rating is included as a measure of the recruiting force. (This measure includes recruiters who are primarily administrators.† While we would have preferred to exclude administrators, data on the number of them are not available for the entire period.)

To adjust for differences in total effort per recruiter, we would prefer to use the exogenously determined quota. Instead, we use actual enlistments (E) because quarterly quota data are unavailable for FY 1972.‡ Since the Navy generally makes goal during the regression period—the shortfall averages 3.5 percent—total enlistment is, with a few exceptions, a close approximation of the quota and, therefore, it, too, is exogenously determined.

Since CY 1975, all the Services have been pressured by Congress and the Department of Defense to increase the quality of enlistments. The Services responded by limiting mental

*The Draft was still in effect during the first six quarters (3Q 1971 - 4Q 1972) used to measure the effects of R and A. In reality, the Services treated this as a tested period for recruiting in an AVF environment. R and A were sharply increased, there were few inductions, and recruiters were pressured through high goals to produce large numbers of enlistments. It seems appropriate, therefore, to measure the effects of R and A with data from these quarters.

†In FY 1977, about 18 percent of the recruiting force were administrators. These recruiters did not receive quotas. Administrators are all recruiters except production recruiters and recruiters in charge of stations having three or less recruiters.

‡We cannot simply drop quarters in FY 1972, because there are too few observations and most of the variation in recruiters occurs that year.

group four enlistments and by increasing recruiters' efforts devoted to enlisting high school graduates. A dummy variable (NAVPL) is included to adjust for the change in high school graduate recruiting effort starting in CY 1975.

Standards

We suspect that changes in standards strongly affected HSGs. Since accurate data on standards are unavailable, we are forced to again use total enlistments as a proxy.* We assume that standards for HSGs declined as the Navy increased its demand for total enlistments. HSGs increased as standards were lowered, because the Navy accepted more HSGs in the lower mental groups and waived various entry requirements for them.

Lacking data on HSG recruit effort and standards applying to them, we use total enlistments as a proxy for both: we assume that as E increases, HSG recruiting effort increases and standards applying to HSGs decline; these changes increase the supply of HSGs. We also use a dummy variable (NAVPL) as a proxy for changes in HSG recruiting effort not captured by E.

We assume HSG supply and demand are in balance in every quarter during the regression period. As demand for total enlistments increase, there is an increase in the demand for HSGs. This is met by increasing the supply of HSGs through increases in recruiting effort and reductions in standards.

Thus, we do not observe a given supply function in the sense that recruiter effort and standards are unchanging throughout the regression period.† Instead, we observe points on a shifting supply function caused by changes in these factors, and shifts are linked to the Navy's total demand for enlistments. By including E and NAVPL we adjust for such changes, and estimate a reduced form equation for high school graduate enlistments.

Population's Awareness and Advertising

Previous studies on advertising have focused on measuring its effects in *product* markets: the consensus is that advertising increases a product's demand in current and future periods, but its effects decline over time.‡

Therefore, we treat Navy advertising as an investment in a stock of intangible "awareness capital" which depreciates over time. This treatment permits its effects to last for more than one period and to decline over time. We estimate the effects of awareness capital on HSGs, and given that, calculate the effect of advertising from the relationship between it and awareness capital.

Construction of a capital stock series for advertising requires that we have data on the advertising effort each quarter, as well as the base period stock and the depreciation rate of awareness capital. Only annual data are available on advertising; lacking quarterly data, we assume the annual advertising budget is spent evenly over the fiscal year. We also assume that

*Another problem is that standards are multi-dimensional. Even if data were available on the many components of standards, increasing the number of variables when there are so few data points would reduce the statistical reliability of the results.

†Here our approach is different from that taken by most previous researchers. They have assumed recruiters' efforts and standards are unchanging.

‡For a review of literature on this issue, see Reference [4].

250 dollars in this quarter's advertising budget (in constant 1967 dollars) generates one unit of awareness capital in the following quarter.*

Because the other data are unavailable we still face serious measurement problems, but estimation procedures are developed to handle them. A way is found to express the base period stock as a regression parameter. This eliminates the need to know it to construct the awareness capital series. A maximum likelihood procedure is used to solve the problem of not knowing the depreciation rate.

Measurement of Awareness Capital

Let K_0 be the base period stock of awareness capital and δ its depreciation rate per quarter. In subsequent periods the net capital stock depends on K_0, δ, and the stream of advertising investments (I_t):

$$K_1 = K_0(1-\delta) + I_0$$
$$K_2 = K_0(1-\delta)^2 + I_0(1-\delta) + I_1$$
$$K_3 = K_0(1-\delta)^3 + I_0(1-\delta)^2 + I_1(1-\delta) + I_2$$
$$\vdots$$
$$K_t = K_0(1-\delta)^t + I_0(1-\delta)^{t-1} + I_1(1-\delta)^{t-2} + \ldots I_{t-1}.$$

The net capital stock in any period t (except for the base period) can be written as the sum of two expressions

$$K_t = K_0 X_t(\delta) + Y_t(\delta) \tag{1}$$

where

$$X_t(\delta) = (1-\delta)^t$$
$$Y_t(\delta) = I_0(1-\delta)^{t-1} + I_1(1-\delta)^{t-2} + \ldots + I_{t-1}.$$

By defining Y_0 equal to zero, Equation (1) also holds in the base period.

Our approach is to include both $X_t(\delta)$ and $Y_t(\delta)$ as explanatory variables in a regression model relating HSGs to various factors. For example, given the regression model

$$\text{dependent variable} = \ldots + a^* K_t + \ldots \tag{2}$$

we substitute for K_t to yield

$$\text{dependent variable} = \ldots + a^* K_0 X_t(\delta) + a^* Y_t(\delta) + \ldots \tag{3}$$

Along with K_0, $X_t(\delta)$ and $Y_t(\delta)$ measure the stock of awareness capital. The coefficient of $Y_t(\delta)$ is the effect on the dependent variable in the regression model of a one unit change in the awareness capital stock.

*Advertising was adjusted for inflation using an overall media price index prepared by the McCann-Erickson advertising agency.

The coefficient of $X_t(\delta)$ is equal to the coefficient of $Y_t(\delta)$ times K_0. Both variables should have a positive sign. An F-test on $X_t(\delta)$ and $Y_t(\delta)$ together is used to test whether awareness capital has a statistically significant effect on HSGs.

Relative Economic Benefits

Four factors are included to measure changes in relative economic benefits.

Pay

The first is the ratio of military to civilian pay. Holding other factors constant, it is expected to have a positive effect on HSGs. The preferred measure involves using the ratio of civilian to military discounted earnings streams. However, assuming long-run earnings growth rates of civilians and enlistees are roughly the same, the measure reduces to simply the ratio of military to civilian pay in a base period. We make this assumption and use as our measure of relative pay the ratio of expected military to expected civilian earnings during an enlistees first term.

Civilian pay is measured by the sum of four-years' full-time, before-tax earnings for a typical 18-21 year old male worker. Similarly, military pay is measured by four-years' before-tax base pay for a typical enlistee. We exclude from military pay in-kind earnings (quarters and subsistance allowances and their tax advantage), since they have hardly changed in real terms over the period.

Unemployment

The second factor is the unemployment rate (UNEMP). As it increases, expected civilian earnings decline and it becomes more difficult and costly to find a civilian job. The unemployment rate of 16-19 year old white males is used to measure unemployment.* An increase in the unemployment rate is expected to increase HSGs.

GI Bill Benefits

Starting in January 1977, GI Bill benefits were reduced and the government no longer paid for all of a veteran's post-service education. Instead, for every dollar the individual pays the government will contribute two.† This change reduced the relative economic benefits of enlisting. A dummy variable starting in January 1977 (GIBL) is included as a factor. It is expected to reduce HSGs.

Ending of the Draft

Although the law authorizing conscription expired on 1 July 1973, for all practical purposes, the Draft ended in January 1973. By eliminating the risk of having to serve in the Army, ending the Draft decreased the benefits of enlisting in the Navy. An All-Volunteer Force dummy variable starting in January 1973 (AVF) is included. Its coefficient is expected to have a negative sign.

*Unemployment rates for other youth cohorts could have been used; but because they tend to move together so closely, we did not experiment with other rates.
†For discussion see Reference [6], p. 27.

Population

The eligible population (POP) is measured by the number of civilian males ages 18-24. (We could use other cohorts, such as 18-19 year olds, but they move together almost perfectly with the 18-24 year old group, so it would not matter statistically which one is chosen).* We expect that an increase in population would increase HSGs.

Seasonal Factors

Seasonal factors such as the ending of the school year affect the quarterly flow of HSGs. We include dummy variables for first, second, and third quarters (Q1, Q2, and Q3) to account for these factors. We expect the third quarter dummy variable, which coincides with the end of the school year, to have a positive coefficient. We have no strong expectations concerning the signs of the other seasonal dummy variables.

Specification of the Model

As suggested by Figure 1, we specify HSGs and MG1-3U HSGs to be nonlinear functions of total enlistments (E)†

$$H = E \sum_i a_i z_i - bE^2 + cE + \nu \tag{4}$$

where H = HSGs or MG1-3U HSGs and the Z_i are the following factors.

PAY	=	Four years' military pay divided by four years' civilian pay
UNEMP	=	Unemployment rate of white males, 16-19 year old
POP	=	18-24 year old male population
GIBL	=	Dummy variable (equal to one for quarters starting in 1977, zero otherwise) measuring effects of declines in GI Bill benefits
NAVPL	=	Dummy variable (equal to one for quarters starting in 1975, zero otherwise) measuring effects of a Navy policy which increased HSG recruiting efforts
AVF	=	Dummy variable (equal to one for quarters starting in 1973, zero otherwise) measuring effects of the ending of the Draft
RECR	=	Number of recruiters
$X_t(\delta)$	=	$(1-\delta)^t$ for δs between 5 and 25 percent
$Y_t(\delta)$	=	$I_0(1-\delta)^{t-1} + I_1 \quad (1-\delta)^{t-2} + \ldots + I_{t-1}$ where I_t is the advertising budget adjusted for inflation in t and δs are between 5 and 25 percent
Q_1	=	Dummy variable (equal to one in first calendar quarter, zero otherwise) measuring seasonality of enlistments
Q_2	=	Dummy variable (equal to one in second calendar quarter, zero otherwise) measuring seasonality of enlistments

*The 18-19 year old group equaled 30.9 percent ± 0.7 percent of the 18-24 year old cohort over the period 1971-1976.
†By using E instead of quotas we still estimate Figure 1. It is shifted slightly to the left, however, because E was 96.5 percent of quotas during the regression period. HSGs and E are nonprior service males excluding reservists.

Q_3	=	Dummy variable (equal to one in the third calendar quarter, zero otherwise) measuring seasonality of enlistments
c	=	Constant term
ν	=	Error term.

This functional form is chosen because it is consistent with two assumptions about the underlying mechanism. The first is that the contribution of each Z_i to the number of HSGs should depend on recruiter effort and HSG recruiting standards, represented by the proxy E. The equation conforms in this respect: the coefficient of Z_i is Ea_i. In terms of *changes* in H, which we may want to bring about through policy:

$$\frac{\partial H}{\partial Z_i} = Ea_i^*/. \tag{5}$$

The second assumption is that the contribution of recruiter effort and standards should show decreasing returns. That is, if recruiters are already aggressive and standards are already low, further changes of this nature should exhibit diminishing returns. The equation conforms because of the negative coefficient of E:

$$\frac{\partial H}{\partial E} = \sum_i a_i Z_i - 2bE + c. \tag{6}$$

Estimation Procedure

It might be inappropriate to apply equation (4) to the data directly.† One of the assumptions of "ordinary least squares" regression is that the error term does not depend systematically on any of the variables. In the present case, however, the error term seems to depend on E; this might lead us to overestimate the variance of parameters. To remove the problem, we divide equation (4) by E.‡

For this reason, the regression equation to be estimated is:

$$\frac{H}{E} = \sum_i a_i Z_i - bE + c + \mu \tag{7}$$

where the error term μ equals ν/E.§

Once the coefficients a_i are obtained, equation (5) is used for estimating R and A productivities.

A search procedure is used to obtain maximum likelihood estimates. Adjusting for autocorrelation using the Cochrane-Orcutt technique,** regression models for HSGs and MG1-3U HSGs are estimated assuming alternative awareness capital depreciation rates per quarter within

*This implies constant marginal effects. We tested this assumption for recruiters.

†Data were obtained from the Navy Recruiting Command, the Defense Manpower Data Center, the McCann-Erickson advertising agency, the Urban Institute, and published sources. For details, see Reference [9], Appendix A.

‡The problem is technically one of "heteroscedasticity." For more discussion, see Reference [17], pp. 259-61.

§Equation (4) was also estimated for HSGs. Result change slightly when equation (7) is used instead of equation (4). While the pattern is the same, some coefficients obtained with equation (4) are about 20 percent larger in absolute value. The greatest difference is in the pay coefficient which is 50 percent larger.

**For discussion, see Reference [17], pp. 277-84.

the relevant range, e.g., in percents 5, 6, . . . , 25. For both HSGs and MG1-3U HSGs, the models that maximized the likelihood function are those for which the depreciation rate is 11 percent.* These best-fit models yield the maximum likelihood estimates of parameters including δ.†

RESULTS

Regression Findings

Maximum likelihood estimates and their t-values are reported in Table 1. A number of factors have statistically significant effects on high school graduate enlistments. For all HSGs, total enlistments and the third quarter dummy variable are significant at the one percent level. The unemployment rate, the number of recruiters, and awareness capital, i.e., $X(11)$ and $Y(11)$ together, are significant at the five percent level. Population and dummy variables GIBL and NAVPL are significant at the ten percent level. PAY, AVF, Q_1, and Q_2 are not statistically significant.‡ The results for MG1-3U HSGs are similar to those for all HSGs.

TABLE 1 — *Regression Findings*

Independent variables	% HSG	t	% MG1-3U	t
CONSTANT	−7.99	−1.60[a]	−3.62	−0.90
PAY	0.655	1.29	0.120	0.28
UNEMP	0.0299	2.64[b]	0.0219	2.02[b]
POP	0.000515	1.63[a]	0.000217	0.84
E	−0.0000108	2.94[c]	−0.0000105	3.30[c]
RECR	0.000244	2.34[b]	0.000215	2.50[b]
X(11)	2.01	1.99[b]	1.18	1.44[a]
Y(11)	0.00000231	1.32[d]	0.000000852	0.58[d]
GIBL	0.0453	1.57[a]	0.00306	0.10
NAVPL	0.0972	1.68[a]	0.153	2.92[c]
AVF	−0.0430	−0.54	0.0273	0.38
Q_1	−0.0307	−1.26	−0.0180	−1.02
Q_2	0.0305	1.56	−0.0144	0.10
Q_3	0.157	3.10[c]	0.105	2.52[b]
$\bar{R}^2$	0.899	N.A.	0.905	N.A.
$F_{(13,11)}$	17.4[c]	N.A.	18.6[c]	N.A.
D-W	2.63	N.A.	2.15	N.A.
Rho	−0.469	2.65[b]	−0.191	−0.97

[a]Statistically significant at ten percent level.
[b]Statistically significant at five percent level.
[c]Statistically significant at one percent level.
[d]F-test of X(11) and Y(11) together indicates statistical significance at five percent level for HSGs and ten percent level for MG1-3U HSGs.

*To derive the likelihood function, it is assumed that, after adjusting for autocorrelation, error terms of the regression model given by equation (7) are independent and normally distributed, having a zero mean and constant variance in each quarter.

†For more discussion on the estimation procedure, see [16] and [17].

‡The result for PAY is probably due to there having been little independent variation: most of the variation consisted of a slight decline toward the end of the regression period when the Navy put a greater emphasis on recruiting HSGs. Thus, the negative effects of relatively small declines in PAY were probably overwhelmed by the positive effects of changes in other variables, e.g., recruiting effort.

The high adjusted R^2s indicate that the models explain most of the variation in the dependent variables. According to the F-statistics, the models are statistically significant at the one percent level. The t-statistics for Rho suggest that adjustments for autocorrelation are perhaps necessary only in the regression model for all HSGs.

Although awareness capital is statistically significant, its impact measured by the coefficient of Y(11) is imprecise.* This may be caused by high collinearity between X(11) and Y(11) (the simple correlation coefficient between them is −0.95).

Except for GIBL, all variables have the expected sign.† The effect is positive for PAY, unemployment, population, NAVPL, recruiters, X(11), Y(11) and Q_3, and negative for total enlistments and AVF.‡

Test of the Model

In measuring the effects of R and A, we assume that advertising operates through a stock of awareness capital having an 11 percent depreciation rate, and that recruiters have an immediate, constant marginal effect. We also estimated the model assuming other treatments of R and A, which yielded evidence supporting our approach and findings.

Treating advertising as a stock rather than a flow improves the fit of equation (7). Changing the depreciation rate reinforces the finding that the effect of advertising is measured imprecisely; however, it has little effect on the recruiter's coefficient or significance level. We also experimented with models that included lagged recruiters and current recruiters squared. Only current recruiters seem to be important, and their marginal productivity appears to be constant over the range of recruiting force levels observed.§

To check the stability of the model, we predicted HSGs in CY 1978 using actual levels of explanatory variables. Predictions are given in Table 2. The model does well: it predicts the sharp decline in HSGs with only a 2.1 percent error for the year. These results increase our confidence in the model and the implied estimates of marginal products for R and A.

The model was estimated with data from a period when HSGs were greatly affected by demand considerations, probably through changes in standards. The model essentially uses goals as a proxy for standards, and in CY 1978 goals were sharply lowered. The predictive

*Imprecise estimates of X(11) and Y(11) may explain a peculiar result obtained for the estimate of the base period capital stock (K_0). It is calculated by the ratio of X(11) and Y(11) coefficients. For HSGs the estimate of K_0 is about 870 thousand units of awareness capital. Although the estimate is positive, as expected, it seems too high. Given a depreciation rate of 11 percent per quarter, to obtain such a high steady-state level would require advertising to be about $24 million per year. Prior to FY 1972, however, advertising was at much lower levels. Since we use the ratio of imprecise estimates to calculate K_0, its estimate is also likely to be imprecise. Perhaps this explains the high estimate of K_0 obtained.

†The unexpected sign for GIBL is probably caused by the fact that a relatively large number of contracts were signed just prior to reduction of G.I. Bill benefits in the first quarter of CY 1977. This caused a spillover of HSGs in 1977, which would tend to reverse the sign for the G.I. Bill quarters — 1Q 1977 - 4Q 1977.

‡The sign of AVF changes direction for MG1-3U HSGs, but it is not statistically significant in either model.

§For details, see Reference [9], Appendix B.

TABLE 2 — *HSG Predictions for CY 1978*

Quarter	Predictions	Actuals	Predictions — actuals	Percent errors
1	9,950	9,880	70	0.7
2	10,440	10,989	−549	−5.0
3	17,731	18,344	−613	−3.3
4	9,631	9,571	60	0.6
Year	47,752	48,784	−1,032	2.1

accuracy of the model suggests that the low level of HSGs achieved in CY 1978 was partly due to increases in standards, and that HSGs might be increased in the future by lowering entry restrictions.

Marginal Products of Recruiters and Advertising

Marginal products (MPs) and elasticities for CY 1978 are given in Table 3. An additional recruiter would yield 14.0 HSGs of which 12.3 would be MG1-3Us. Thus, most of the recruiters' impact is on HSGs in the upper mental groups. These enlistments are generated immediately, based on our experiments with other models including lagged recruiters.

TABLE 3 — *Effects of Recruiters and Advertising in CY 1978*[a]

Supply factors	All HSGs (48,784)		MG1-3U HSGs (29,844)[b]	
	MP	Elasticity	MP	Elasticity
One recruiter				
All (3425)	14.0	0.98	12.3	1.42
Except administrators (2808)	14.0	0.80	12.3	1.16
$1M advertising[c]				
($10 million)	1206	0.25	445	0.15

[a]Numbers in parentheses are CY 1978 levels of variables which are used to calculate elasticities.
[b]Estimated from ratio of MG1-3U HSGs to all HSGs in first three quarters of CY 1978, e.g., 0.61.
[c]Effect of $1 million of advertising.

The recruiters elasticity is 0.98 for HSGs, but if we exclude administrators it is only 0.80. The lower elasticity is probably more accurate. This is because the regression model yields an estimate for the type of recruiter added during the regression period, and we suspect that few of those added were administrators.

A once-and-for-all increase in the advertising budget of $1 million would yield 1,206 HSGs in the long-run. Only 445 or 37 percent would be in the upper mental groups, however.*

The effects of advertising are felt mostly in the future. Based on the estimated depreciation rate of 11 percent per quarter, we calculate that only 30 percent of the total impact is felt this year; it takes four years for 82 percent to accrue.

*For details on how the effects of advertising are calculated, see Reference [9], Appendix C.

Thus, R and A both affect HSGs, but the effects differ in two important respects which make advertising less attractive: unlike recruiters, advertising predominately affects HSGs in the lower mental groups; and, rather than this year, most of its effects are felt in the future.

Recruiter and Advertising Cost Per HSG

Given marginal productivities and estimates of marginal costs, we calculate the cost of generating HSGs with R and A. These are given in Table 4. As discussed earlier, a $1 million increase in advertising in 1967 dollars would yield 1,206 HSGs in the long-run. Given the change in advertising costs since 1967, in CY 1978 it would take $2.2 million of advertising to produce what $1 million could buy in 1967. The ratio of $2.2 million to 1,206 HSGs or $1,824 is advertising's marginal cost for HSGs. The cost per MG1-3U HSG is $4,944.

The cost of an additional recruiter in CY 1978 was about $31,000. Given the recruiter productivities in Table 3, the marginal cost of enlistments is $2,214 for HSGs and $2,520 for MG1-3U HSGs.

Although in the long run, the cost per HSG is slightly lower for advertising, in the short run it is more costly than recruiters. This is because only 30 percent of the HSGs generated by advertising accrue in the first year. As a result, the cost per HSG generated in the first year is about three times higher than recruiters. To calculate it, we divide $2.2 million by the HSGs generated by advertising in the first year (362) to yield a cost of $6,077 per HSG.

TABLE 4 — *Long-Run Cost Per HSG in CY 1978 for One Recruiter and $1 Million of Advertising*

Group	$1 Million advertising			One recruiter		
	Marginal productivity	Marginal cost[a] (millions)	Cost per HSG	Marginal productivity	Marginal cost[b]	Cost per HSG
All HSGs	1,206	$2.2	$1,824	14.0	$31,000	$2,214
MG1-3U HSGs	445	$2.2	$4,944	12.3	$31,000	$2,520

[a]The marginal cost of $1 million of advertising equals the advertising price index times $1 million. The price index is forecast to be 2.2 in CY 1978, based on the 1977 price level (2.02) adjusted upward by the 1977 rate of inflation (9 percent).

[b]Marginal cost includes: $19,000 for recruiter's salary, allowances, retirement, training, reenlistment bonus, and special pay; and $12,000 for support costs including salaries and operations and maintenance expenses.

In summary, the long-run cost of generating HSGs is about the same for R and A — $2,000 per HSG. For MG1-3U HSGs in the long or short run and for all HSGs in the short run, it is less costly to use recruiters.

CONCLUSIONS

Our findings suggest that R and A could be used to increase high school graduate enlistments. But, since advertising affects HSGs mostly in the lower mental groups, the mix of additional R and A should be weighted more heavily towards recruiters.

With regard to temporary shortfalls, additional resources are needed that would have an immediate effect. Since about half of the HSGs generated by advertising accrue after the second year, the proper strategy is to add just recruiters.

Changes in the unemployment rate have an important effect on HSGs: a one percentage point increase in the youth unemployment rate would yield about 1,700 HSGs and 1250 would be in the upper mental groups; elasticities are 0.46 for HSGs and 0.55 for MG1-3U HSGs. Because of its impact and tendency to change frequently, unemployment should be taken into account when formulating near-term R and A requirements.

The recruiting difficulty of the Navy in FY 1978-79 was probably caused by declines in unemployment and relative pay; another factor might have been increases in standards. Still, the shortfall of HSGs has been a modest 10 percent. It can probably be met with just modest increases in R and A and slight reductions in standards.

In the 1980s, however, the shortfall would be larger than 10 percent. There will be a reduction in the youth population which will not only reduce the eligible population, it is also likely to lead to lower unemployment rates and higher civilian wages. These changes would further reduce the supply of HSGs. Therefore, future shortfalls are likely to be larger, and to meet them may require increases in military pay and benefits in addition to the remedies previously suggested.

Regarding methodology, measurement of the productivity of advertising is difficult because (1) advertising seems to have only slight effects which are distributed over time, and (2) HSGs are strongly affected by other supply factors as well as by demand considerations. These problems were overcome by treating advertising as an investment in awareness capital, and by including it and numerous other factors in a reduced form regression model.

While our methodology is interesting, the data are extremely crude and estimates are based on only 26 observations. Therefore, results and policy recommendations should be viewed as highly tentative. More research is needed to provide a solid basis for R and A policy making.

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